### Chapter Overview and Pacing

**LESSON OBJECTIVES**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Block</th>
<th>Basic/ Average</th>
<th>Advanced</th>
<th>Regular</th>
<th>Basic/ Average</th>
<th>Advanced</th>
</tr>
</thead>
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<tr>
<td><strong>14-1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Graphing Trigonometric Functions</strong> <em>(pp. 762–768)</em></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
| *• Graph trigonometric functions.  
• Find the amplitude and period of variation of the sine, cosine, and tangent functions.* | | | | | | |
| **14-2** | | | | | | |
| **Translations of Trigonometric Graphs** *(pp. 769–776)*  | | | | | | |
| *• Graph horizontal translations of trigonometric graphs and find phase shifts.  
• Graph vertical translations of trigonometric graphs.* | | | | | | |
| **14-3** | | | | | | |
| **Trigonometric Identities** *(pp. 777–781)*  | | | | | | |
| *• Use identities to find trigonometric values.  
• Use trigonometric identities to simplify expressions.* | | | | | | |
| **14-4** | | | | | | |
| **Verifying Trigonometric Identities** *(pp. 782–785)*  | | | | | | |
| *• Verify trigonometric identities by transforming one side of an equation into the form of the other side.  
• Verify trigonometric identities by transforming each side of the equation into the same form.* | | | | | | |
| **14-5** | | | | | | |
| **Sum and Difference of Angles Formulas** *(pp. 786–790)*  | | | | | | |
| *• Find values of sine and cosine involving sum and difference formulas.  
• Verify identities by using sum and difference formulas.* | | | | | | |
| **14-6** | | | | | | |
| **Double-Angle and Half-Angle Formulas** *(pp. 791–797)*  | | | | | | |
| *• Find values of sine and cosine involving double-angle formulas.  
• Find values of sine and cosine involving half-angle formulas.* | | | | | | |
| **14-7** | | | | | | |
| **Solving Trigonometric Equations** *(pp. 798–804)*  | | | | | | |
| *Preview: Solving Trigonometric Equations  
• Solve trigonometric equations.  
• Use trigonometric equations to solve real-world problems.* | | | | | | |
| **Study Guide and Practice Test** *(pp. 805–809)*  | | | | | | |
| **Standardized Test Practice** *(pp. 810–811)*  | | | | | | |
| **Chapter Assessment**  | | | | | | |
| **TOTAL** | | 0 | 14 | 0 | 7.5 | 7.5 |

*Pacing suggestions for the entire year can be found on pages T20–T21.*
### Chapter Resource Manager

**CHAPTER 14 RESOURCE MASTERS**

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<th>Practice (Skills and Average)</th>
<th>Reading to Learn Mathematics</th>
<th>Enrichment</th>
<th>Assessment</th>
<th>Applications</th>
<th>5-Minute Check Transparencies</th>
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<td>GCS 54</td>
<td>14-3</td>
<td>14-3</td>
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<td>861–862</td>
<td>863–864</td>
<td>865</td>
<td>866</td>
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<td></td>
<td>14-5</td>
<td>14-5</td>
<td></td>
</tr>
<tr>
<td>867–868</td>
<td>869–870</td>
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<td>872</td>
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<td>873–874</td>
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<td>878</td>
<td>894</td>
<td>SC 28, SM 145–148</td>
<td>14-7</td>
<td>14-7</td>
<td>(Preview: graphing calculator)</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>879–892, 896–898</td>
</tr>
</tbody>
</table>

*Key to Abbreviations: GCS = Graphing Calculator and Spreadsheet Masters, SC = School-to-Career Masters, SM = Science and Mathematics Lab Manual*
This chapter continues the extensive investigation of trigonometric functions from the previous chapter. This lesson focuses on graphs of the trigonometric functions. Each of the sine, cosine, and tangent functions repeats a pattern of values, or is periodic. For the sine and cosine functions, the period is \(2\pi\) radians or 360°, while for the tangent function the period is \(\pi\) radians or 180°. For periodic functions the distance between a horizontal center line and the maximum or minimum value is called the amplitude of the graph. For \(y = \sin x\) and \(y = \cos x\), the horizontal center line is the x-axis and the maximum and minimum values are ±1, so the amplitude is 1.

The lesson also describes these properties algebraically. For the sine function \(y = a \sin b\theta\) and the cosine function \(y = a \cos b\theta\), the period is \(2\pi / |b|\) and the amplitude is \(|a|\). For \(y = a \tan b\theta\), the tangent function, the period is \(\pi / |b|\). The tangent function has no finite maximum or minimum, so amplitude is not defined for the tangent function.

In this lesson students explore the graphs of \(y = \sin \theta\) and \(y = \cos \theta\). More specifically, they use the functions \(y = a \sin b(\theta - h) + k\) and \(y = a \cos b(\theta - h) + k\) and see how changing each of the values \(a\), \(b\), \(h\), and \(k\) affects the graph. Also, they explore how to sketch a graph for a given set of values of the four variables.

One change in the graph of a periodic function is to move the horizontal center line of the graph. When \(k = 0\) the horizontal center line is the x-axis and the vertical shift is zero. A positive value of \(k\) represents a vertical shift upward while a negative value of \(k\) represents a downward vertical shift. The amplitude is determined by the value of \(|a|\), so the maximum values of the function are \(|a|\) units above \(k\) and the minimum values of the function are \(|a|\) units below \(k\). A third change is the period of the function. The expression for the length of one period has the variable \(b\) in the denominator, so as the value of \(|b|\) increases the period of the function decreases. The fourth variable, \(h\), is associated with the phase shift of the function. If \(h\) is positive, the entire graph is shifted to the right; if \(h\) is negative, the entire graph is shifted to the left. Students also use the equation \(y = a \tan b(\theta - h) + k\) and explore phase shifts, periods, and vertical shifts for the tangent function.
14-3 Trigonometric Identities

This lesson and the next three deal with trigonometric identities. Students learn the definition of an identity, and they work with arguments that are half of a given angle, twice a given angle, or the sum or difference of two given angles. In this lesson students work with the definitions of the six trigonometric functions in terms of \( x, y, \) and \( r \). By dividing each side of \( x^2 + y^2 = r^2 \) by \( r^2, y^2, \) or \( x^2, \) the results are three identities called the Pythagorean Identities. For other identities, called the Reciprocal Identities, students note that the definitions for sine and cosecant, for cosine and secant, and for tangent and cotangent are reciprocals. Also, they see that the ratios \( \sin \theta \div \cos \theta \) and \( \cos \theta \div \sin \theta \) can be simplified to \( \tan \theta \) and \( \cot \theta \), respectively, resulting in two identities called the Quotient Identities. Students explore how to use identities to simplify trigonometric expressions, and they use identities to evaluate a complicated trigonometric expression for a given argument.

14-4 Verifying Trigonometric Identities

In this lesson students continue exploring how to identify and use trigonometric identities. For each equation, the goal is to transform each side, replacing expressions with equivalent expressions, until the two sides are identical. There are several approaches for writing equivalent expressions. First, students can make substitutions using the Pythagorean Identities. Second, they can use the Distributive Property to factor an expression or to collect like terms. Third, they can transform a term by multiplying the term by an expression equivalent to 1. And fourth, they can rewrite all the trigonometric functions in terms of \( \sin \theta \) or \( \cos \theta \) by using the Quotient and Reciprocal Identities. Students also relate trigonometric identities to graphs, using a graphing calculator to show that the expressions on each side of a trigonometric identity have the same graph.

14-5 Sum and Difference of Angles Formulas

Students derive and then use formulas for rewriting the two-variable functions \( \sin (\alpha \pm \beta) \) and \( \cos (\alpha \pm \beta) \) in terms of the one-variable functions \( \sin \alpha, \sin \beta, \cos \alpha, \) and \( \cos \beta \). The derivation of the difference formula for the cosine function begins with the two ordered pairs on the unit circle that correspond to two angles \( \alpha \) and \( \beta \). The distance \( d \) between the two points can be found using the distance formula, or it can be found as the distance between the point \((1, 0)\) and the coordinates of the point on the unit circle associated with angle \( (\alpha - \beta) \). After equating the two expressions for \( d \), algebraic manipulation gives an expression for the two-variable function \( \cos (\alpha - \beta) \) in terms of one-variable functions. Students use the formulas to find exact values for particular trigonometric expressions. They also use the formulas in problems such as verifying that the equation \( \sin (180^\circ + \theta) = -\sin \theta \) is an identity.

14-6 Double-Angle and Half-Angle Formulas

Students begin with the formulas for \( \sin (\alpha + \beta) \) and \( \cos (\alpha + \beta) \) and replace both \( \alpha \) and \( \beta \) with \( \theta \). The results, called the Double-Angle Formulas, are equations in which each of \( \sin 2\theta \) and \( \cos 2\theta \) is expressed in terms of \( \sin \theta \) and \( \cos \theta \). Then students use an algebraic technique and let \( \alpha \) represent \( 2\theta \) (so \( \frac{\alpha}{2} \) represents \( \theta \)), and derive formulas for \( \sin \frac{\alpha}{2} \) and \( \cos \frac{\alpha}{2} \) in terms of \( \sin \alpha \) and \( \cos \alpha \). The two formulas are called the Half-Angle Formulas. Students use the Half-Angle and Double-Angle Formulas, along with other formulas, to find exact values for particular trigonometric expressions. They also substitute the formulas in equations to verify trigonometric identities.

14-7 Solving Trigonometric Equations

In this last lesson of the two-chapter investigation of trigonometric functions, students solve trigonometric equations and review some of the important general ideas of algebra. The first step in solving a trigonometric equation is to use factoring, the zero product property, and identities to rewrite a complicated equation as a string of simpler trigonometric equations. The second step is to use trigonometric inverses to isolate the variable; that is, to solve an equation such as \( \cos \theta = 0.5 \) for \( \theta \). The third step is to use ideas of periodicity to include all the occurrences of that value. Students solve trigonometric equations for arguments measured in degrees or in radians, and they use trigonometric equations and their solutions to solve statements of real-world problems.
### Additional Intervention Resources

The Princeton Review’s *Cracking the SAT & PSAT*

The Princeton Review’s *Cracking the ACT*

ALEKS

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### TestCheck and Worksheet Builder

This networkable software has three modules for intervention and assessment flexibility:

- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.
Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

**Student Edition**
- Foldables Study Organizer, p. 761
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 766, 774, 779, 784, 788, 794, 802, 805)
- Writing in Math questions in every lesson, pp. 768, 776, 781, 785, 790, 796, 804
- Reading Study Tip, pp. 786, 788
- WebQuest, pp. 775, 804

**Teacher Wraparound Edition**
- Foldables Study Organizer, pp. 761, 805
- Study Notebook suggestions, pp. 766, 774, 779, 783, 788, 794, 802
- Modeling activities, pp. 768, 790
- Speaking activities, pp. 781, 784, 803
- Writing activities, pp. 776, 797
- **ELL Resources**, pp. 760, 767, 775, 780, 785, 789, 796, 803, 805

**Additional Resources**
- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 14 Resource Masters*, pp. vii–viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 14 Resource Masters*, pp. 841, 847, 853, 859, 865, 871, 877)
- *Vocabulary PuzzleMaker* software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- *Reading and Writing in the Mathematics Classroom*
- *WebQuest and Project Resources*

For more information on **Reading and Writing in Mathematics**, see pp. T6–T7.

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**Intervention Technology**

**Alge2PASS: Tutorial Plus** CD-ROM offers a complete, self-paced algebra curriculum.

<table>
<thead>
<tr>
<th>Algebra 2 Lesson</th>
<th>Alge2PASS Lesson</th>
</tr>
</thead>
<tbody>
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<td>14-2</td>
<td>27 <em>Graphing Trigonometric Functions</em></td>
</tr>
<tr>
<td>14-4</td>
<td>28 <em>Trigonometric Identities</em></td>
</tr>
</tbody>
</table>

**Intervention at Home**

**Log on for student study help.**

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes. [www.algebra2.com/extra_examples](http://www.algebra2.com/extra_examples)
  [www.algebra2.com/self_check_quiz](http://www.algebra2.com/self_check_quiz)
- For chapter review, there is vocabulary review, test practice, and standardized test practice. [www.algebra2.com/vocabulary_review](http://www.algebra2.com/vocabulary_review) [www.algebra2.com/chapter_test](http://www.algebra2.com/chapter_test) [www.algebra2.com/standardized_test](http://www.algebra2.com/standardized_test)

For more information on Intervention and Assessment, see pp. T8–T11.
Have students read over the list of objectives and make a list of any words with which they are not familiar.

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Some equations contain one or more trigonometric functions. It is important to know how to simplify trigonometric expressions to solve these equations. Trigonometric functions can be used to model many real-world applications, such as music. You will learn how a trigonometric function can be used to describe music in Lesson 14-6.

Key to NCTM Standards:
1=Number & Operations, 2=Algebra, 3=Geometry, 4=Measurement,
5=Data Analysis & Probability, 6=Problem Solving, 7=Reasoning & Proof,
8=Communication, 9=Connections, 10=Representation

Vocabulary Builder
The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the Chapter 14 Resource Masters. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 14 test.
### Prerequisite Skills
To be successful in this chapter, you’ll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 14.

**For Lessons 14-1 and 14-2**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \sin 135^\circ = \frac{\sqrt{2}}{2} )</td>
</tr>
<tr>
<td>2.</td>
<td>( \tan 315^\circ = -1 )</td>
</tr>
<tr>
<td>3.</td>
<td>( \cos 90^\circ = 0 )</td>
</tr>
<tr>
<td>4.</td>
<td>( \tan 45^\circ = 1 )</td>
</tr>
<tr>
<td>5.</td>
<td>( \sin \frac{5\pi}{4} = \frac{\sqrt{2}}{2} )</td>
</tr>
<tr>
<td>6.</td>
<td>( \cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>7.</td>
<td>( \sin \frac{11\pi}{6} = -\frac{1}{2} )</td>
</tr>
<tr>
<td>8.</td>
<td>( \tan \frac{3\pi}{2} ) not defined</td>
</tr>
</tbody>
</table>

**For Lessons 14-3, 14-5, and 14-6**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.</td>
<td>( \cos (-150^\circ) = -\frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>10.</td>
<td>( \sin 510^\circ = \frac{1}{2} )</td>
</tr>
<tr>
<td>11.</td>
<td>( \cot \frac{9\pi}{4} = 1 )</td>
</tr>
<tr>
<td>12.</td>
<td>( \sec \frac{13\pi}{6} = \frac{2\sqrt{3}}{3} )</td>
</tr>
<tr>
<td>13.</td>
<td>( \tan \left(-\frac{3\pi}{2}\right) ) not defined</td>
</tr>
<tr>
<td>14.</td>
<td>( \csc (-720^\circ) ) not defined</td>
</tr>
<tr>
<td>15.</td>
<td>( \cos \frac{7\pi}{3} = -\frac{1}{2} )</td>
</tr>
<tr>
<td>16.</td>
<td>( \tan \frac{8\pi}{3} = -\sqrt{3} )</td>
</tr>
</tbody>
</table>

**For Lesson 14-4**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Factor Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.</td>
<td>(-15x^2 - 5x - 5x(3x + 1)) prime</td>
</tr>
<tr>
<td>18.</td>
<td>(2x^4 - 4x^2 - 2x^2(x^2 - 2))</td>
</tr>
<tr>
<td>19.</td>
<td>(x^3 + 4) prime</td>
</tr>
<tr>
<td>20.</td>
<td>(x^2 - 6x + 8 \ (x - 4)(x - 2))</td>
</tr>
<tr>
<td>21.</td>
<td>(2x^2 - 3x - 2 \ (2x + 1)(x - 2))</td>
</tr>
<tr>
<td>22.</td>
<td>(3x^2 - 2x^2 - x \ 3x(x + 1)(x - 1))</td>
</tr>
</tbody>
</table>

**For Lesson 14-7**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solve Quadratic Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.</td>
<td>(x^2 - 5x - 24 = 0 \ 8, -3)</td>
</tr>
<tr>
<td>24.</td>
<td>(x^2 - 2x - 48 = 0 \ 8, -6)</td>
</tr>
<tr>
<td>25.</td>
<td>(x^2 + 3x - 40 = 0 \ -8, 5)</td>
</tr>
<tr>
<td>26.</td>
<td>(x^2 - 12x = 0 \ 0, 12)</td>
</tr>
<tr>
<td>27.</td>
<td>(-2x^2 - 11x - 12 = 0 \ -4, -\frac{3}{2})</td>
</tr>
<tr>
<td>28.</td>
<td>(x^2 - 16 = 0 \ -4, 4)</td>
</tr>
</tbody>
</table>

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**Foldables Study Organizer**

Make this Foldable to help you organize information about trigonometric graphs and identities. Begin with eight sheets of grid paper.

**Step 1** Staple
Staple the stack of grid paper along the top to form a booklet.

**Step 2** Cut and Label
Cut seven lines from the bottom of the top sheet, six lines from the second sheet, and so on. Label with lesson numbers as shown.

**Reading and Writing**
As you read and study the chapter, use each page to write notes and to graph examples for each lesson.

---

**Writing Instructions and Sequencing Data**

After students make their Foldable, have them label each tab to correspond to a lesson in this chapter. Students use their Foldable to take notes, define terms, record concepts, and write examples. After each lesson, ask students to write a set of instructions on how to do something presented in the lesson. For example, a student might write instructions for graphing trigonometric functions. Have students follow their own instructions to check them for accuracy. Use their notes and textbook to make needed revisions.

---

**Getting Started**

This section provides a review of the basic concepts needed before beginning Chapter 14. Page references are included for additional student help.

**Prerequisite Skills** in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Prerequisite Skill</th>
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</tr>
<tr>
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<td>Properties of Equality (p. 781)</td>
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<td>Simplifying Radical Expressions (p. 785)</td>
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<tr>
<td>14-6</td>
<td>Solving Equations Using the Square Root Property (p. 790)</td>
</tr>
<tr>
<td>14-7</td>
<td>Solving Equations Using the Zero Product Property (p. 797)</td>
</tr>
</tbody>
</table>
Graphing Trigonometric Functions

What You’ll Learn

• Graph trigonometric functions.
• Find the amplitude and period of variation of the sine, cosine, and tangent functions.

Why can you predict the behavior of tides?

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.

Graph Trigonometric Functions

In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is called a periodic function.

To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

To graph the periodic functions \( y = \sin \theta \), \( y = \cos \theta \), or \( y = \tan \theta \), use values of \( \theta \) expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form \((\theta, \sin \theta), (\theta, \cos \theta), \) and \((\theta, \tan \theta), \) respectively.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>225°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>315°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>( -1 )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>( -\frac{1}{2} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>nearest tenth</td>
<td>0</td>
<td>0.5</td>
<td>0.7</td>
<td>0.9</td>
<td>1</td>
<td>0.9</td>
<td>0.7</td>
<td>0.5</td>
<td>0</td>
<td>-0.5</td>
<td>-0.7</td>
<td>-0.9</td>
<td>-1</td>
<td>-0.9</td>
<td>-0.7</td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>( -\frac{\sqrt{2}}{2} )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>-1</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>( -\frac{1}{2} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>nearest tenth</td>
<td>1</td>
<td>0.9</td>
<td>0.7</td>
<td>0.5</td>
<td>0</td>
<td>-0.5</td>
<td>-0.7</td>
<td>-0.9</td>
<td>-1</td>
<td>-0.9</td>
<td>-0.7</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
<td>0.7</td>
<td>0.9</td>
<td>1</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>nd</td>
<td>-( \sqrt{3} )</td>
<td>-1</td>
<td>( -\frac{\sqrt{3}}{3} )</td>
<td>0</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>nd</td>
<td>-( \sqrt{3} )</td>
<td>-1</td>
<td>( -\frac{\sqrt{3}}{3} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>nearest tenth</td>
<td>0.6</td>
<td>1</td>
<td>1.7</td>
<td>nd</td>
<td>-1.7</td>
<td>-1</td>
<td>-0.6</td>
<td>0</td>
<td>0.6</td>
<td>1</td>
<td>1.7</td>
<td>nd</td>
<td>-1.7</td>
<td>-1</td>
<td>-0.6</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\( \theta \) = 0, \( \frac{\pi}{6} \), \( \frac{\pi}{4} \), \( \frac{\pi}{3} \), \( \frac{\pi}{2} \), \( \frac{2\pi}{3} \), \( \frac{3\pi}{4} \), \( \frac{\pi}{2} \), \( \frac{5\pi}{6} \), \( \pi \), \( \frac{7\pi}{6} \), \( \frac{5\pi}{4} \), \( \frac{3\pi}{2} \), \( \frac{5\pi}{3} \), \( \frac{7\pi}{4} \), \( \frac{11\pi}{6} \), \( 2\pi \)

nd = not defined
After plotting several points, complete the graphs of \( y = \sin \theta \) and \( y = \cos \theta \) by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or \( 2\pi \) radians. That is, the graph of each function repeats itself every 360° or \( 2\pi \) radians.

Notice that both the sine and cosine have a maximum value of 1 and a minimum value of \(-1\). The amplitude of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is \( \frac{1 - (-1)}{2} \) or 1.

The graphs of the tangent function can also be drawn by plotting points. By examining the values for \( \tan \theta \) in the table, you can see that the tangent function is not defined for \( 90^\circ, 270^\circ, \ldots, 90^\circ + k \cdot 180^\circ \), where \( k \) is an integer. The graph is separated by vertical asymptotes whose \( x \)-intercepts are the values for which \( y = \tan \theta \) is not defined.

The period of the tangent function is 180° or \( \pi \) radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

The graphs of the secant, cosecant, and cotangent functions are shown below. Compare them to the graphs of the cosine, sine, and tangent functions, which are shown in red.

Notice that the period of the secant and cosecant functions is 360° or \( 2\pi \) radians. The period of the cotangent is 180° or \( \pi \) radians. Since none of these functions have a maximum or minimum value, they have no amplitude.
Concept Check

After discussing the Key Concept box about amplitudes and periods, ask: What is an equation involving the sine function with a period of 90° and an amplitude of \( \frac{1}{2} \)? One of the following:

\[ y = \frac{1}{2} \sin 4\theta, \ y = \frac{1}{2} \sin (-4\theta), \ y = -\frac{1}{2} \sin 4\theta, \text{ or } y = -\frac{1}{2} \sin (-4\theta) \]

Amplitude and Period

Note that the amplitude affects the graph along the vertical axis and the period affects it along the horizontal axis.

7. When \( a \) is positive, the amplitude is \( a \). When \( a \) is negative, then amplitude is \( |a| \). \( a \) has no effect on the period.

Key Concept

Amplitudes and Periods

- **Words**
  
  For functions of the form \( y = a \sin b\theta \) and \( y = a \cos b\theta \), the amplitude is \( |a| \), and the period is \( \frac{360°}{|b|} \) or \( \frac{\pi}{|b|} \).

  For functions of the form \( y = a \tan b\theta \), the amplitude is not defined, and the period is \( \frac{180°}{|b|} \) or \( \frac{\pi}{|b|} \).

- **Examples**
  
  \[ y = 3 \sin 4\theta \quad \text{amplitude 3 and period } \frac{360°}{4} \text{ or } 90° \]
  
  \[ y = -6 \cos 5\theta \quad \text{amplitude } |-6| \text{ or 6 and period } \frac{2\pi}{5} \]
  
  \[ y = 2 \tan \frac{3}{4}\theta \quad \text{no amplitude and period } 3\pi \]

Graphing Calculator Investigation

Graphing Trigonometric Functions

To set the calculator for degrees, press **MODE** and move the cursor to highlight **DEGREE** and press **ENTER**. Also, be sure to have students clear the Y= lists before beginning Exercise 1.
You can use the amplitude and period of a trigonometric function to help you graph the function.

**Example 1** Graph Trigonometric Functions

Find the amplitude and period of each function. Then graph the function.

a. \( y = \cos 3\theta \)

First, find the amplitude.

\[ |a| = 1 \quad \text{The coefficient of } \cos 3\theta \text{ is } 1. \]

Next, find the period.

\[ \frac{360^\circ}{b} = \frac{360^\circ}{3} = 120^\circ \]

Use the amplitude and period to graph the function.

b. \( y = \frac{1}{4} \sin \theta \)

Amplitude: \( |a| = \frac{1}{4} \)

\[ = \frac{1}{4} \]

Period: \( \frac{360^\circ}{b} = \frac{360^\circ}{1} = 360^\circ \)

![Graph of \( y = \cos 3\theta \)]

![Graph of \( y = \frac{1}{4} \sin \theta \)]

![Graph of \( y = \frac{1}{2} \sin \left( -\frac{1}{3}\theta \right) \)]

b. \( y = \frac{1}{3} \cos \theta \)

Amplitude: \( |a| = \frac{1}{3} \)

\[ = \frac{1}{3} \]

Period: \( \frac{2\pi}{b} = \frac{2\pi}{-\frac{1}{3}} = 6\pi \)

![Graph of \( y = \frac{1}{3} \cos \theta \)]

![Graph of \( y = \frac{1}{2} \sin \left( -\frac{1}{3}\theta \right) \)]

![Figure 1](https://www.algebra2.com/extra_examples)

![Figure 2](https://www.algebra2.com/extra_examples)

**Teacher to Teacher**

Berchie Holliday  
Author, Cincinnati, OH

“I have my students construct a unit circle on a coordinate plane with a toothpick length radius. They mark every 15°. Students form right triangles inside the circle and break toothpicks to match the lengths of each vertical leg. (See Figure 1 at the right.) They transfer each leg to its appropriate degree mark on a second x-axis and place a dot at the top of each toothpick. (See Figure 2.) Finally, students connect the dots with a smooth curve.”


In-Class Example

2 Oceanography

Refer to the application at the beginning of the lesson. The tidal range in the Bay of Fundy in Canada measures 50 feet.

a. Write a function to represent the height \( h \) of the tide. Assume that the tide is at equilibrium at \( t = 0 \) and that the high tide is beginning.

\[
y = 25 \sin \frac{\pi}{6} t
\]

b. Graph the tide function.

![Graph of tide function](image)

Practice/Apply

3

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Find the error

Students should quickly notice that Dante must be incorrect because his graph does not have an amplitude of 3. Ask students if they can identify the mistake that Dante made.

More About...

Oceanography

Lake Superior has one of the smallest tidal ranges. It can be measured in inches, while the tidal range in the Bay of Fundy in Canada measures up to 50 feet.

Source: Office of Naval Research

Concept Check

1. Open ended Explain why \( y = \tan \theta \) has no amplitude. See margin.

2. Explain what it means to say that the period of a function is 180°.

3. Find the error Dante and Jamile graphed \( y = 3 \cos \frac{2 \pi}{3} \theta \).

Who is correct? Explain your reasoning.

Answer

1. Sample answer: Amplitude is half the difference between the maximum and minimum values of a graph; \( y = \tan \theta \) has no maximum or minimum value.
Find the amplitude, if it exists, and period of each function. Then graph each function. 15–32. See pp. 811A–811N.

4. \( y = \frac{1}{2} \sin \theta \)
5. \( y = 2 \sin \theta \)
6. \( y = \frac{2}{3} \cos \theta \)
7. \( y = \frac{1}{4} \tan \theta \)
8. \( y = \csc 2\theta \)
9. \( y = 4 \sin 2\theta \)
10. \( y = 4 \cos \frac{3}{4} \theta \)
11. \( y = \frac{1}{2} \sec 3\theta \)
12. \( y = \frac{3}{4} \cos \frac{1}{2} \theta \)

**Application**

**BIOLOGY** For Exercises 13 and 14, use the following information.

In a certain wildlife refuge, the population of field mice can be modeled by \( y = 3000 + 1250 \sin \frac{\pi}{6} t \), where \( y \) represents the number of mice and \( t \) represents the number of months past March 1 of a given year.

13. Determine the period of the function. What does this period represent?
14. What is the maximum number of mice and when does this occur? 4250; June 1

**Practice and Apply**

**Homework Help**

Find the amplitude, if it exists, and period of each function. Then graph each function. 13–14. See pp. 811A–811N.

- For Exercises 15–32, see pp. 811A–811N for graphs; \( y = \frac{3}{4} \sin 4\theta \).
- For Exercises 33–40, see pp. 811A–811N for graphs; \( y = 7 \cos 5\theta \).

**More About**

**MEDICINE**

Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

36. \( y = 0.25 \sin 128\pi t \)
37. \( y = 0.25 \sin 512\pi t \).
38. \( f(x) = \cos x \) and \( f(x) = \sec x \). See pp. 811A–811N for graphs.

**Exercises Examples**

Find the amplitude, if it exists, and period of each function. Then graph each function. 15–32. See pp. 811A–811N.

15. \( y = 3 \sin \theta \)
16. \( y = 5 \cos \theta \)
17. \( y = 2 \csc \theta \)
18. \( y = 2 \tan \theta \)
19. \( y = \frac{1}{3} \sin \theta \)
20. \( y = \frac{1}{3} \sec \theta \)
21. \( y = \sin 4\theta \)
22. \( y = \sin 2\theta \)
23. \( y = \sec 3\theta \)
24. \( y = \cot 5\theta \)
25. \( y = 4 \tan \frac{1}{2} \theta \)
26. \( y = 2 \cot \frac{1}{2} \theta \)
27. \( y = 6 \sin \frac{3}{2} \theta \)
28. \( y = 3 \cos \frac{3}{2} \theta \)
29. \( y = 3 \csc \frac{1}{2} \theta \)
30. \( y = \frac{1}{2} \cot 2\theta \)
31. \( y = \cot \theta \)
32. \( y = \frac{3}{4} \sin \frac{3}{2} \theta \)

**Skills Practice, p. 839 and Practice, p. 840**

Find the amplitude, if it exists, and period of each function. Then graph each function.

- \( y = -0.5 \sin 4\theta \)
- \( y = -0.5 \cos 4\theta \)
- \( y = -0.5 \tan 4\theta \)

**Reading to Learn Mathematics, p. 841**

**Answer**

37. Sample answer: The amplitudes are the same. As the frequency increases, the period decreases.

**Enrichment, p. 842**

**Blueprints**

Integrating blueprints requires the ability to select and use trigonometric functions and geometric properties. The figure below represents a plan for an improvement to a roof. The existing fitting shown makes a 30° angle with the horizontal. The inclined fitting is placed at an angle of 45° from the horizontal. If the existing fitting is perpendicular to the roof, use the appropriate trigonometric function to find the unknown measurement.

**Name**

Find the unknown measurement in the figure at the right. The dimensions are scaled.

1. The length of the hypotenuse: \( a = 10 \), \( b = 10 \), \( c = 10 \).
2. The length of the horizontal component: \( a = 5 \), \( b = 5 \), \( c = 5 \).
3. The length of the vertical component: \( a = 5 \), \( b = 5 \), \( c = 5 \).

**Reading Improvement**

1. The length of the hypotenuse is: \( a = 10 \), \( b = 10 \), \( c = 10 \).
2. The length of the horizontal component is: \( a = 5 \), \( b = 5 \), \( c = 5 \).
3. The length of the vertical component is: \( a = 5 \), \( b = 5 \), \( c = 5 \).

**Enrichment**

2. What is the change in elevation of the plan? \( 45^\circ \) change.
3. What is the change in slope of the plan? \( 45^\circ \) change.
4. What is the change in slope of the plan? \( 45^\circ \) change.
5. What is the change in slope of the plan? \( 45^\circ \) change.
6. What is the change in slope of the plan? \( 45^\circ \) change.

**Helping You Remember**

2. What is the change in elevation of the plan? \( 45^\circ \) change.
3. What is the change in slope of the plan? \( 45^\circ \) change.
4. What is the change in slope of the plan? \( 45^\circ \) change.
5. What is the change in slope of the plan? \( 45^\circ \) change.
6. What is the change in slope of the plan? \( 45^\circ \) change.
About the Exercises...
Organization by Objective
- Graph Trigonometric Functions: 15–34
- Variations of Trigonometric Functions: 15–34

Odd/Even Assignments
Exercises 15–34 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide
Basic: 15–31 odd, 35–56
Average: 15–35 odd, 36–56
Advanced: 16–34 even, 36–52 (optional: 53–56)

4 Assess
Open-Ende Assessment
Modeling Provide students with a coordinate grid and a length of string. Give students one of the six basic trigonometric functions and have them model the graph using the string. Have students check their model with a graphing calculator.

Getting Ready for Lesson 14-2
PREREQUISITE SKILL Students will graph horizontal and vertical translations of trigonometric functions in Lesson 14-2. Students will apply what they learned about families of quadratic functions. Use Exercises 53–56 to determine your students’ familiarity with the graphs of families of functions.

Answers
40. About the Exercises...  

BOATING For Exercises 39–41, use the following information.  
A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

39. Write an equation for the motion of the buoy. Assume that it is at equilibrium at t = 0 and that it is on the way up from the normal water level.  
\[ y = 2 \sin \left( \frac{\pi}{5} t \right) \]

40. Draw a graph showing the height of the buoy as a function of time. See margin.

41. What is the height of the buoy after 12 seconds? about 1.9 ft

42. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. See margin.

Why can you predict the behavior of tides? Include the following in your answer:
- an explanation of why certain tidal characteristics follow the patterns seen in the graph of the sine function, and
- a description of how to determine the amplitude of a function using the maximum and minimum values.

43. What is the period of \( f(x) = \frac{1}{2} \cos 3x \)? A
   \[
   \begin{align*}
   \text{A} & \quad 120^\circ \\
   \text{B} & \quad 180^\circ \\
   \text{C} & \quad 360^\circ \\
   \text{D} & \quad 720^\circ
   \end{align*}
   \]

44. Identify the equation of the graphed function. C
   \[
   \begin{align*}
   \text{A} & \quad y = \frac{1}{2} \sin 4\theta \\
   \text{B} & \quad y = 2 \sin \frac{1}{4}\theta \\
   \text{C} & \quad y = \frac{1}{4} \sin 2\theta \\
   \text{D} & \quad y = 4 \sin \frac{1}{2}\theta
   \end{align*}
   \]

Maintain Your Skills
Mixed Review
Solve each equation. (Lesson 13-7)
45. \( x = \sin^{-1} 1 \) 90°
46. \( \arcsin (-1) = y \) -90°
47. \( \arccos \frac{\sqrt{2}}{2} = x \) 45°

Find the exact value of each function. (Lesson 13-6)
48. \( \sin 300° = \frac{1}{2} \)
49. \( \sin (-315°) = \frac{\sqrt{2}}{2} \)
50. \( \cos 405° = \frac{\sqrt{2}}{2} \)

51. PROBABILITY There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. (Lesson 12-5)
   \[
   \frac{3}{16}
   \]

52. Find the first five terms of the sequence in which \( a_1 = 3, a_{n+1} = 2a_n + 5 \). (Lesson 11-5)
   \[
   \{3, 11, 27, 59, 123\}
   \]

Getting Ready for the Next Lesson
PREREQUISITE SKILL Graph each pair of functions on the same set of axes. (To review graphs of quadratic functions, see Lesson 6-6.)
53. \( y = x^2, y = 3x^2 \)
54. \( y = 3x^2, y = 3x^2 - 4 \)
55. \( y = 2x^2, y = (x + 1)^2 \)
56. \( y = x^2 + 2, y = (x - 3)^2 + 2 \)

42. Sample answer: Tides display periodic behavior. This means that their pattern repeats at regular intervals. Answers should include the following information.
   - Tides rise and fall in a periodic manner, similar to the sine function.
   - In \( f(x) = a \sin bx \), the amplitude is the absolute value of \( a \).
Translating Trigonometric Graphs

**What You'll Learn**
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

**Vocabulary**
- phase shift
- vertical shift
- midline

**How can translations of trigonometric graphs be used to show animal populations?**

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population $R$ can be modeled by the equation

$$R = 1200 + 250 \sin \frac{\pi}{6} t,$$

where $t$ is the time in years since January 1, 2001.

**HORIZONTAL TRANSLATIONS** Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

**Graphing Calculator Investigation**

**Horizontal Translations**

On a TI-83 Plus, set the MODE to degrees.

**Think and Discuss** 1–3. See margin.

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?
2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?
3. What conjecture can you make about the effect of $h$ in the function $y = \sin (x - h)$?
4. Test your conjecture on the following pairs of graphs.
   - $y = \cos x$ and $y = \cos (x + 30)$
   - $y = \tan x$ and $y = \tan (x - 45)$
   - $y = \sec x$ and $y = \sec (x + 75)$

Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If $(x, y)$ are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \pm h)$. A horizontal translation of a trigonometric function is called a phase shift.

Lesson 14-2 Translations of Trigonometric Graphs 769
In Lesson 6-6, students learned about the translations of graphs of quadratic functions. In this lesson, students will use similar techniques to study the translations of graphs of the trigonometric functions.

**HORIZONTAL TRANSLATIONS**

**In-Class Example**

1. State the amplitude, period, and phase shift for each function. Then graph the function.
   
   a. \( y = 2 \sin (\theta + 20^\circ) \) amplitude: 2; period: 360°; phase shift: 20° left
   
   ![Graph of \( y = 2 \sin (\theta + 20^\circ) \)]

   b. \( y = \frac{1}{2} \cos (\theta - \frac{\pi}{6}) \) amplitude: \( \frac{1}{2} \); period: 2\( \pi \); phase shift: \( \frac{\pi}{6} \) right
   
   ![Graph of \( y = \frac{1}{2} \cos (\theta - \frac{\pi}{6}) \)]

**Answers (p. 769)**

**Graphing Calculator Investigation**

1. The graph of \( y = \sin (x - 30) \) is shifted 30° to the right of the graph of \( y = \sin x \).

2. The graph of \( y = \sin (x + 60) \) is shifted 60° to the left of the graph of \( y = \sin x \).

3. Sample answer: When \( h \) is positive the graph shifts right \( h \) units. When \( h \) is negative the graph shifts left \( h \) units.

**Graphing Calculator Investigation**

**Graphing the Secant Function** The graph of the secant function should look like a pattern of U shapes, alternately opening upward and downward. Students might want to extend the activity by investigating the graphs of cosecant (csc) and cotangent (cot). The graphing calculator does not have a button for graphing either of these functions so students should enter \( Y = \frac{1}{\sin x} \) to graph the cosecant function, and they should enter \( Y = \frac{1}{\tan x} \) to graph the cotangent function.
VERTICAL TRANSLATIONS In Chapter 6, you learned that the graph of \( y = x^2 + 4 \) is a vertical translation of the parent graph of \( y = x^2 \). Similarly, graphs of trigonometric functions can be translated vertically through a vertical shift.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If \( (x, y) \) are coordinates of \( y = \sin x \), then \( (x, y \pm k) \) are coordinates of \( y = \sin x \pm k \).

A new horizontal axis called the midline becomes the reference line about which the graph oscillates. For the graph of \( y = \sin \theta + k \), the midline is the graph of \( y = k \).

### Key Concept

**Vertical Shift**

<table>
<thead>
<tr>
<th>Words</th>
<th>Vertical Shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>The vertical shift of the functions ( y = a \sin b(\theta - h) + k ), ( y = a \cos b(\theta - h) + k ), and ( y = a \tan b(\theta - h) + k ) is ( k ). If ( k &gt; 0 ), the shift is up. If ( k &lt; 0 ), the shift is down. The midline is ( y = k ).</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Models</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = a \sin b(\theta - h) + k ), ( k &gt; 0 )</td>
<td>( y = a \cos b(\theta - h) + k ), ( k &gt; 0 )</td>
<td>( y = a \tan b(\theta - h) + k ), ( k &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>( y = a \sin b(\theta - h) + k ), ( k &lt; 0 )</td>
<td>( y = a \cos b(\theta - h) + k ), ( k &lt; 0 )</td>
<td>( y = a \tan b(\theta - h) + k ), ( k &lt; 0 )</td>
<td></td>
</tr>
</tbody>
</table>

The secant, cosecant, and cotangent can be graphed using the same rules.

### Example 2: Graph Vertical Translations

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

**a.** \( y = 2 \sin \theta - 1 \)
- **vertical shift:** \(-1\); **midline:** \( y = -1 \); **amplitude:** 2; **period:** \( 2\pi \)

**b.** \( y = \frac{1}{2} \cos \theta + 3 \)
- **vertical shift:** 3; **midline:** \( y = 3 \); **amplitude:** \( \frac{1}{2} \); **period:** \( 2\pi \)
Example 3

The function is written in the form \( y = a \cos[b(\theta - h)] + k \). Identify the values of \( a \), \( b \), and \( h \).

\( k = -6 \), so the vertical shift is \(-6\).
\( a = 4 \), so the amplitude is \( |4| = 4 \).
\( b = \frac{1}{2} \), so the period is \( 2\pi \) or \( 4\pi \).
\( h = \frac{\pi}{3} \), so the phase shift is \( \frac{\pi}{3} \) to the right.

Then graph the function.

**Step 1** The vertical shift is \(-6\). Graph the midline \( y = -6 \).

**Step 2** The amplitude is 4. Draw dashed lines 4 units above and below the midline at \( y = -2 \) and \( y = -10 \).

**Step 3** The period is \( 4\pi \), so the graph will be stretched. Graph \( y = 4 \cos\frac{1}{2}\theta - 6 \) using the midline as a reference.

**Step 4** Shift the graph \( \frac{\pi}{3} \) to the right.
Example 4 Use Translations to Solve a Problem

Suppose a person’s resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person’s resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure for \( t \) seconds. Then graph the function.

**Explore** You know that the function is periodic and can be modeled using sine.

**Plan** Let \( P \) represent blood pressure and let \( t \) represent time in seconds. Use the equation \( P = a \sin [b(t - h)] + k \).

**Solve**
- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.
  \[
P = \frac{120 + 80}{2} \quad \text{or} \quad 100
\]
- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.
  \[
a = |120 - 100| \quad a = |80 - 100| \\
= 20 \quad \text{or} \quad 20
\]
  Thus, \( a = 20 \).
- Determine the period of the function and solve for \( b \). Recall that the period of a function can be found using the expression \( \frac{2\pi}{|b|} \). Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.
  \[
1 = \frac{2\pi}{|b|} \quad \text{Write an equation.}
\]
  \[
|b| = 2\pi \quad \text{Multiply each side by } |b|.
\]
  \[
b = \pm 2\pi \quad \text{Solve.}
\]
  For this example, let \( b = 2\pi \). The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).
- There is no phase shift, so \( h = 0 \). So, the equation is \( P = 20 \sin 2\pi t + 100 \).
- Graph the function.
  
  **Step 1** Draw the midline
  \[
P = 100.
\]
  **Step 2** Draw maximum and minimum reference lines.
  **Step 3** Use the period to draw the graph of the function.
  **Step 4** There is no phase shift.

**Examine** Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.

---

**Differentiated Instruction**

**Kinesthetic** Make coordinate axes with masking tape on the classroom floor. Give students at least 15 feet of rope and have them stand along the \( x \)-axis, positioning the rope to model the graph of \( y = \sin x \). As you call out equations of functions whose graphs are horizontal phase shifts of the graph of \( y = \sin x \), students can step left or right to model the translated graph. Similarly, call out functions whose graphs are vertical shifts of the graph of \( y = \sin x \).
3. Sample answer: \( y = \sin(\theta + 45^\circ) \)
State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

25. \( y = \sin \theta - 1; y = -1; 1; y = 360^\circ \)
26. \( y = \sec \theta + 2; y = 2; \) no amplitude; \( 360^\circ \)
27. \( y = \cos \theta - 5; y = -5; 1; y = 360^\circ \)
28. \( y = \csc \theta - \frac{3}{2}; y = \frac{3}{2}; y = 360^\circ \)
29. \( y = \frac{1}{2} \sin \theta + 1; y = \frac{1}{2}; 2; y = 360^\circ \)
30. \( y = 6 \cos \theta + 1.5; y = 1.5; 6; 360^\circ \)

31. Graph \( y = 5 + \tan (\theta + \frac{\pi}{4}) \). Describe the transformation to the parent graph \( y = \tan \theta \). See margin for graph; translation \( \frac{\pi}{4} \) units left and 5 units up.

Draw a graph of the function \( y = \frac{2}{3} \cos (\theta - 50^\circ) + 2 \). How does this graph compare to the graph of \( y = \cos \theta \)? See margin for graph; translation 50° right and 2 units up with an amplitude of \( \frac{2}{3} \) unit.

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

33–42. See pp. 811A–811N.

33. \( y = 2 \sin (3(\theta - 45^\circ)) + 1 \)
34. \( y = 4 \cos (2(\theta + 30^\circ)) - 5 \)
35. \( y = 3 \csc \left(\frac{1}{2}(\theta + 60^\circ)\right) - 3.5 \)
36. \( y = 6 \cot \left(\frac{3}{2}(\theta - 90^\circ)\right) + 0.75 \)
37. \( y = \frac{1}{4} \cos (2(\theta - 150^\circ)) + 1 \)
38. \( y = \frac{2}{5} \tan (6(\theta + 135^\circ)) - 4 \)
39. \( y = 3 + 2 \sin \left(\frac{2}{3}(\theta + \frac{\pi}{3})\right) \)
40. \( y = 4 + 5 \sec \left(\frac{3}{2}(\theta + \frac{2\pi}{3})\right) \)

41. Graph \( y = 3 - \frac{3}{2} \cos \theta \) and \( y = 3 + \frac{1}{2} \cos (\theta + \pi) \). How do the graphs compare?

42. Compare the graphs of \( y = -\sin \left(\frac{1}{4}(\theta - \frac{3\pi}{2})\right) \) and \( y = \cos \left(\frac{1}{4}(\theta + 3\pi)\right) \).

43. **MUSIC** When represented on an oscilloscope, the note A above middle C has period of \( \frac{1}{440} \). Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is \( K \).
- a. \( y = K \sin 220\pi t \)
- b. \( y = K \sin 440\pi t \)
- c. \( y = K \sin 880\pi t \)

44. Find the maximum number of owls. After how many years does this occur?
45. What is the minimum number of mice? How long does it take for the population of mice to reach this level?
46. Why would the maximum owl population follow behind the population of mice? See margin.
47. **TIDES** The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height \( h \) of the water \( t \) hours after noon on the first day.

**Online Research Data Update** Use the Internet or another resource to find tide data for a location of your choice. Write a sine function to represent your data. Then graph the function. Visit www.algebra2.com/data_update to learn more.

**Enrichment, p. 848**

Translating Graphs of Trigonometric Functions

These graphs are drawn at the right.
- \( y = 3 \sin (\theta - 30^\circ) \)
- \( y = 3 \sin (\theta - 60^\circ) \)

Replacing \( (x, y) \) with \( (x - 30^\circ, y) \) translates the graph to the right. Replacing \( (x, y) \) with \( (x - 60^\circ, y) \) translates the graph left.

Step 1: Graph one cycle of \( y = 4 \cos (\theta - 80^\circ) + 2 \).

Step 2: Translate the function to the new location.

Drawing the Triangular Graphs of Trigonometric Functions

Step 1: Translating the graph of the function \( y = \cos \theta \\ preserve in the map.

Step 2: Translating the graph of the function \( y = \sin \theta \\ preserve in the map.

Helping You Remember

Many students have trouble remembering which of the functions \( y = \sin \theta \) and \( y = \cos \theta \) represents a shift to the left and which represents a shift to the right. Using \( -\theta \), explains a good way to remember which is which.

Sample answer: Although sine curves are infinitely repeating periodic graphs, think of \( y = \sin \theta \) starting a period or cycle at \((0, 0)\). Then \( y = \sin \theta \\ preserve in the map. There is a shift to the left at \((0, 0)\), a shift of \( 90^\circ \) to the left, while \( y = \sin \theta \\ preserve in the map. Starts later at \((0, 0)\), a shift of \( 90^\circ \) to the right.

**Reading to Learn Mathematics, p. 847**

**ELL**

**Pre-Activity** How can translations of trigonometric graphs be used to show animal populations?

Refer to the next page.

Reading the Lesson

1. Determine whether the graph of each function represents a shift in the parent function.
- a. \( y = \sin \theta - 60^\circ \)
- b. \( y = \sin (\theta - 60^\circ) \)
- c. \( y = \sin (\theta + 60^\circ) - 60^\circ \)

2. Determine whether the graph of each function has an amplitude change, period change, phase shift, or vertical shift compared to the graph of the parent function. Only one of these may apply to each function. Do not actually graph the functions.
- a. \( y = \sin \theta \\ preserve in the map. \)
- b. \( y = \cos (\theta + 60^\circ) \)
- c. \( y = \sin (\theta + 60^\circ) - 60^\circ \)

3. Describe the effect on the graph of each function. Do not actually graph the functions.
- a. \( y = \sin \theta \\ preserve in the map. \)
- b. \( y = \cos (\theta + 60^\circ) \)
- c. \( y = \sin (\theta + 60^\circ) - 60^\circ \)

**Helping You Remember**

Many students have trouble remembering which of the functions \( y = \sin \theta \) and \( y = \cos \theta \) represents a shift to the left and which represents a shift to the right. Using \( -\theta \), explains a good way to remember which is which.

Sample answer: Although sine curves are infinitely repeating periodic graphs, think of \( y = \sin \theta \) starting a period or cycle at \((0, 0)\). Then \( y = \sin \theta \\ preserve in the map. There is a shift to the left at \((0, 0)\), a shift of \( 90^\circ \) to the left, while \( y = \sin \theta \\ preserve in the map. Starts later at \((0, 0)\), a shift of \( 90^\circ \) to the right.
48. **CRITICAL THINKING** The graph of \( y = \cot \theta \) is a transformation of the graph of \( y = \tan \theta \). Determine \( a, b, \) and \( h \) so that \( \cot \theta = a \tan [(\theta - h)] \) for all values of \( \theta \) for which each function is defined. \( a = -1, \ b = 1, \ h = \frac{\pi}{2} \)

49. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See pp. 811A–811N.

How can translations of trigonometric graphs be used to show animal populations?

Include the following in your answer:
- a description of what each number in the equation \( R = 1200 + 250 \sin \left( \frac{1}{2} \pi t \right) \) represents, and
- a comparison of the graphs of \( y = a \cos bx, \ y = a \cos bx + k, \) and \( y = a \cos [b(x - h)] \).

50. Which equation is represented by the graph? **B**

- \( y = \cot (\theta + 45^\circ) \)
- \( y = \cot (\theta - 45^\circ) \)
- \( y = \tan (\theta + 45^\circ) \)
- \( y = \tan (\theta - 45^\circ) \)

51. Identify the equation for a sine function of period \( 90^\circ \), after a phase shift \( 20^\circ \) to the left. **D**

- \( y = \sin [0.25(\theta - 20^\circ)] \)
- \( y = \sin [4(\theta - 20^\circ)] \)
- \( y = \sin [0.25(\theta + 20^\circ)] \)
- \( y = \sin [4(\theta + 20^\circ)] \)

52. amplitude: does not exist; period: \( 360^\circ \) or \( 2\pi \)

53. amplitude: 1; period: \( 720^\circ \) or \( 4\pi \)

54. amplitude: does not exist; period: \( 270^\circ \) or \( \frac{3\pi}{2} \)
Trigonometric Identities

**What You’ll Learn**
- Use identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

**Vocabulary**
- trigonometric identity

**How can trigonometry be used to model the path of a baseball?**

A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of \( v \) feet per second at an angle of \( \theta \) from the horizontal, the height \( h \) of the ball after \( t \) seconds can be represented by

\[
h(t) = \left( \frac{-16}{v^2 \cos^2 \theta} \right) t^2 + \left( \frac{\sin \theta}{\cos \theta} \right) t + h_0,
\]

where \( h_0 \) is the height of the ball in feet the moment it is hit.

**Find Trigonometric Values**

In the equation above, the second term \( \left( \frac{\sin \theta}{\cos \theta} \right) t \) can also be written as \( (\tan \theta)t \). \( \left( \frac{\sin \theta}{\cos \theta} \right) t = (\tan \theta)t \) is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) is true except for angle measures such as \( 90^\circ, 270^\circ, 450^\circ, \ldots, 90^\circ + 180^\circ \cdot k \). The cosine of each of these angle measures is 0, so none of the expressions \( \tan 90^\circ, \tan 270^\circ, \tan 450^\circ \), and so on, are defined. An identity similar to this is \( \cot \theta = \frac{\cos \theta}{\sin \theta} \).

These identities are sometimes called **quotient identities**. These and other basic trigonometric identities are listed below.

<table>
<thead>
<tr>
<th>Key Concept</th>
<th>Basic Trigonometric Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quotient Identities</strong></td>
<td>( \tan \theta = \frac{\sin \theta}{\cos \theta} )</td>
</tr>
<tr>
<td><strong>Reciprocal Identities</strong></td>
<td>( \csc \theta = \frac{1}{\sin \theta} )</td>
</tr>
<tr>
<td><strong>Pythagorean Identities</strong></td>
<td>( \cos^2 \theta + \sin^2 \theta = 1 )</td>
</tr>
</tbody>
</table>

You can use trigonometric identities to find values of trigonometric functions.

**Lesson 14-3  Trigonometric Identities  777**
FIND TRIGONOMETRIC VALUES

**Teaching Tip** You can use the familiar definitions of sine, cosine, and tangent as the ratios of the opposite side, adjacent side, and hypotenuse of a right triangle to show why $\frac{\sin \theta}{\cos \theta} = \tan \theta$.

**In-Class Example**

1. Find tan $\theta$ if sec $\theta = -2$ and $180^\circ < \theta < 270^\circ$. $\tan \theta = \sqrt{3}$
2. Find sin $\theta$ if cos $\theta = -\frac{1}{\sqrt{2}}$ and $90^\circ < \theta < 180^\circ$. $\sin \theta = \frac{\sqrt{3}}{2}$

SIMPLIFY EXPRESSIONS

**In-Class Example**

2. Simplify $\sin \theta (\csc \theta - \sin \theta)$. $\cos^2 \theta$

**Example 1** Find a Value of a Trigonometric Function

a. Find $\cos \theta$ if $\sin \theta = -\frac{3}{5}$ and $90^\circ < \theta < 180^\circ$.

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 & \text{Trigonometric identity} \\
\cos^2 \theta &= 1 - \sin^2 \theta & \text{Subtract $\sin^2 \theta$ from each side.} \\
\cos^2 \theta &= 1 - \left(\frac{3}{5}\right)^2 & \text{Substitute $\frac{3}{5}$ for $\sin \theta$.} \\
\cos^2 \theta &= 1 - \frac{9}{25} & \text{Square $\frac{3}{5}$.} \\
\cos^2 \theta &= \frac{16}{25} & \text{Subtract.} \\
\cos \theta &= \frac{4}{\sqrt{25}} & \text{Take the square root of each side.} \\
\cos \theta &= \frac{4}{5} & \text{Since $\theta$ is in the second quadrant, $\cos \theta$ is negative. Thus, $\cos \theta = -\frac{4}{5}$.}
\end{align*}
\]

b. Find $\csc \theta$ if $\cot \theta = -\frac{1}{\sqrt{2}}$ and $270^\circ < \theta < 360^\circ$.

\[
\begin{align*}
\cot^2 \theta + 1 &= \csc^2 \theta & \text{Trigonometric identity} \\
\left(-\frac{1}{\sqrt{2}}\right)^2 + 1 &= \csc^2 \theta & \text{Substitute $-\frac{1}{\sqrt{2}}$ for $\cot \theta$.} \\
\frac{1}{2} + 1 &= \csc^2 \theta & \text{Square $-\frac{1}{\sqrt{2}}$.} \\
\frac{3}{2} &= \csc^2 \theta & \text{Add.} \\
\frac{3}{2} &= \csc^2 \theta & \text{Add.} \\
\frac{3}{2} &= \csc^2 \theta & \text{Take the square root of each side.} \\
\csc \theta &= \frac{\sqrt{3}}{\sqrt{2}} & \text{Since $\theta$ is in the fourth quadrant, $\csc \theta$ is negative. Thus, $\csc \theta = -\frac{\sqrt{3}}{2}$.
\end{align*}
\]

**Example 2** Simplify an Expression

\[
\begin{align*}
\csc^2 \theta - \cot^2 \theta &= \frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\cos^2 \theta} \\
&= \frac{1 - \cos^2 \theta}{\sin^2 \theta} & \text{Add.} \\
&= \frac{\sin^2 \theta}{\cos^2 \theta} & \text{Subtract $\cos^2 \theta$ from the numerator.} \\
&= \frac{\sin \theta \cdot \sin \theta}{\cos \theta} & \text{Simplify.} \\
&= \frac{\sin \theta \cdot \sin \theta}{\cos \theta} & \text{Add.} \\
&= \frac{1}{\cos \theta} & \text{Take the square root of each side.}
\end{align*}
\]

**Differentiated Instruction**

**Logical** Have students work in groups of three. Ask each group to choose one of the identities in the Key Concept box on p. 777 and work together to demonstrate that it is true. Students should verify their results using the definitions of sine, cosine, and tangent in terms of the sides of a right triangle.
**Example 3** Simplify and Use an Expression

**BASEBALL** Refer to the application at the beginning of the lesson. Rewrite the equation in terms of \( \tan \theta \).

\[
h = \left( \frac{-16}{v^2 \cos^2 \theta} \right)^2 + \left( \sin \theta \cos \theta \right) t + h_0 \quad \text{Original equation}
\]

\[
= \frac{-16}{v^2} \left( \frac{1}{\cos^2 \theta} \right)^2 + \left( \sin \theta \cos \theta \right) t + h_0 \quad \text{Factor.}
\]

\[
= \frac{-16}{v^2} \left( \sec^2 \theta \right)^2 + \left( \tan \theta \right) t + h_0 \quad \sin \theta = \tan \theta
\]

\[
= \frac{-16}{v^2} \left( \sec^2 \theta \right)^2 + \left( \tan \theta \right) t + h_0 \quad \text{Since } \sec^2 \theta = 1 - \tan^2 \theta.
\]

\[
= \frac{-16}{v^2} (1 + \tan^2 \theta)^2 + (\tan \theta) t + h_0
\]

Thus, \( \left( \frac{-16}{v^2 \cos^2 \theta} \right)^2 + \left( \sin \theta \cos \theta \right) t + h_0 = \frac{-16}{v^2} (1 + \tan^2 \theta)^2 + (\tan \theta) t + h_0 \).

---

**Check for Understanding**

**Concept Check**

1–3. See margin.

1. Describe how you can determine the quadrant in which the terminal side of angle \( \alpha \) lies if \( \sin \alpha = -\frac{3}{4} \).

2. Explain why the Pythagorean identities are so named.

3. OPEN ENDED Explain what it means to simplify a trigonometric expression.

**Guided Practice**

Find the value of each expression.

4. \( \tan \theta \), if \( \sin \theta = \frac{1}{2}; 90^\circ \leq \theta < 180^\circ \), \( -\frac{\sqrt{3}}{3} \)

5. \( \csc \theta \), if \( \cos \theta = \frac{3}{5}; 180^\circ \leq \theta < 270^\circ \), \( \frac{5}{3} \)

6. \( \cos \theta \), if \( \sin \theta = \frac{4}{5}; 0^\circ \leq \theta < 90^\circ \), \( \frac{3}{5} \)

7. \( \sec \theta \), if \( \tan \theta = -1; 270^\circ < \theta < 360^\circ \), \( \frac{\sqrt{2}}{2} \)

8. \( \csc \theta \cos \theta \tan \theta \)

9. \( \sec^2 \theta - 1 \tan^2 \theta \)

10. \( \frac{\tan \theta}{\sin \theta} \sec \theta \)

11. \( \sin \theta \left( 1 + \cot^2 \theta \right) \csc \theta \)

**Application**

12. **PHYSICAL SCIENCE** When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the angle of inclination and is represented by the equation \( \tan \theta = \frac{v^2 - h^2}{gR} \), where \( R \) is the radius of the circular path, \( v \) is the speed of the person in meters per second, and \( g \) is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using \( \sin \theta \) and \( \cos \theta \).

\[
\sin \theta = \cos \theta \frac{v^2}{gR}
\]

**Practice and Apply**

Find the value of each expression.

13. \( \tan \theta \), if \( \cot \theta = 2; 0^\circ \leq \theta < 90^\circ \), \( \frac{1}{2} \)

14. \( \sin \theta \), if \( \cos \theta = \frac{2}{3}; 0^\circ \leq \theta < 90^\circ \), \( \frac{\sqrt{5}}{3} \)

15. \( \sec \theta \), if \( \tan \theta = -2; 90^\circ < \theta < 180^\circ \)

16. \( \tan \theta \), if \( \sec \theta = -3; 180^\circ < \theta < 270^\circ \)

---

**Answers**

1. Sample answer: The sine function is negative in the third and fourth quadrants. Therefore, the terminal side of the angle must lie in one of those two quadrants.

2. Sample answer: Pythagorean identities are derived by applying the Pythagorean Theorem to trigonometric concepts.

3. Sample answer: Simplifying a trigonometric expression means writing the expression as a numerical value or in terms of a single trigonometric function, if possible.
Chapter 14 

Simplify each expression.

**Find Trigonometric Values**

Write an identity that you could use to find each of the indicated trigonometric values and tell whether that value is positive or negative. (Do not actually find the values.)

- a. \( \cot 2 \sin \)
- b. \( \csc 5 \theta \)
- c. \( \sec 4 \theta \)
- d. \( \cot 3 \theta \sin \)
- e. \( \tan 7 \theta \)
- f. \( \sin 8 \theta \)

**Identities**

- -3

**Trigonometric Functions**

- a. \( \sin 5 \theta \) and \( \cos 3 \theta \)
- b. \( \tan 7 \theta \) and \( \sin 8 \theta \)
- c. \( \sec 4 \theta \) and \( \cot 3 \theta \)
- d. \( \csc 5 \theta \) and \( \tan 2 \theta \)
- e. \( \sqrt{\sin^2 \theta + \cos^2 \theta} \) and \( \cot \theta \)
- f. \( \sqrt{\sin^2 \theta + \cos^2 \theta} \) and \( \csc \theta \)

**AMUSEMENT PARKS**

The oldest operational carousel in the United States is the Flying Horse Carousel at Martha’s Vineyard, Massachusetts. Source: Martha’s Vineyard Preservation Trust

**LITRITILL**

For Exercises 40 and 41, use the following information.

The amount of light that a source provides to a surface is called the illuminance. The illuminance \( E \) in foot candles on a surface is related to the distance \( R \) in feet from the light source. The formula

\[ E = \frac{l \cos \theta}{R^2} \]

**Electronics**

For Exercises 42 and 43, use the following information.

When an alternating current of frequency \( f \) and a peak current \( I \) pass through a circuit, the power delivered to the resistance \( R \) is given by

\[ P = IR - I^2R \cos \theta \]

41. Is the equation in Exercise 40 equivalent to \( R^2 = \frac{l \tan \theta \cos \theta}{E} \)? Explain.

**Critical Thinking**

If \( \tan \beta = \frac{3}{4} \) find \( \sin \beta \) sec \( \beta \) and \( \tan \beta \).

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45. **Writing in Math**  
Answer the question that was posed at the beginning of the lesson. See margin.

How can trigonometry be used to model the path of a baseball?

Include the following in your answer:

- an explanation of why the equation at the beginning of the lesson is the same as $y = -16 \sec^2 \theta x^2 + (\tan \theta) x + h_0$ and
- examples of how you might use this equation for other situations.

46. If $\sin x = m$ and $0 < x < 90^\circ$, then $\tan x = B$

\[ \frac{1}{m^2} . \]

\[ \frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} = A \]

\[ 2 \sec^2 x \]

\[ - \sec^2 x \]

\[ 2 \csc^2 x \]

\[ - \csc^2 x \]

47. **Maintain Your Skills**

**Mixed Review**  
State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. (Lesson 14-2)

48. $y = \sin \theta - 1$; $y = -1$; $1$; $360^\circ$

49. $y = \tan \theta + 12$

12; $y = 12$; no amplitude; $180^\circ$

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1) 50–52. See pp. 811A–811N.

50. $y = \csc 2\theta$

51. $y = \cos 3\theta$

52. $y = \frac{1}{3} \cot 5\theta$

53. Find the sum of a geometric series for which $a_1 = 48, a_n = 3$, and $r = \frac{1}{2}$. (Lesson 14-4) 93

54. Write an equation of a parabola with focus at $(11, -1)$ and whose directrix is $y = 2$. (Lesson 14-2) $y = -\frac{1}{6}(x - 11)^2 + \frac{1}{2}$

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** Name the property illustrated by each statement. (To review properties of equality, see Lesson 1-3.)

55. Symmetric (=)

56. If $7 + s = 21$, then $s = 14$. Subt. (=)

57. If $4x = 16$, then $12x = 48$. Multiplication (=)

58. If $q + (8 + 5) = 32$, then $q + 13 = 32$.

**Practice Quiz 1** The quiz provides students with a brief review of the concepts and skills in Lessons 14-1 through 14-3. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

**Answers**

48.

49.
14-4 Verifying Trigonometric Identities

What You'll Learn
- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

How can you verify trigonometric identities?
Examine the graphs of $y = \tan^2 \theta - \sin^2 \theta$ and $y = \tan^2 \theta \sin^2 \theta$. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of $\theta$ and conclude that the statement is true for all values of $\theta$. In order to show that the equation $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ for all values of $\theta$, you must consider the general case.

**TRANSFORM ONE SIDE OF AN EQUATION**
You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity, you need to show that it is true for all values of $\theta$.

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation separately until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

**Example 1 Transform One Side of an Equation**
Verify that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity.

Transform the left side.

\[
\begin{align*}
\tan^2 \theta - \sin^2 \theta &= \frac{\tan^2 \theta}{\cos^2 \theta} - \sin^2 \theta \\
&= \frac{\tan^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \\
&= \frac{\tan^2 \theta - \sin^2 \theta}{\cos^2 \theta} \\
&= \frac{\sin^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \\
&= \frac{\tan^2 \theta \sin^2 \theta}{\cos^2 \theta} \\
&= \tan^2 \theta \sin^2 \theta
\end{align*}
\]

Thus, $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity.
Example 2  Find an Equivalent Expression

Multiple-Choice Test Item

\[ \sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\cot \theta} \right) = \]

A  \cos \theta  \quad B  \sin \theta  \quad C  \cos^2 \theta  \quad D  \sin^2 \theta

Read the Test Item

Find an expression that is equal to the given expression.

Solve the Test Item

Write a trigonometric identity by using the basic trigonometric identities and the definitions of trigonometric functions to transform the given expression to match one of the choices.

\[
\sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\cot \theta} \right) = \sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\frac{\cos \theta}{\sin \theta}} \right) = \sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta \cdot \sin \theta}{\cos \theta} \right) = \sin \theta \left( \frac{1}{\sin \theta} - \sin \theta \right) = \sin \theta \left( \frac{1 - \sin^2 \theta}{\sin \theta} \right) = \cos^2 \theta.
\]

Since \( \sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\cot \theta} \right) = \cos^2 \theta \), the answer is C.

Example 3  Verify by Transforming Both Sides

Verify that \( \sec^2 \theta - \tan^2 \theta = \tan \theta \cot \theta \) is an identity.

\[
\frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos \theta} = \frac{\sin \theta \cdot \cos \theta}{\sin \theta \cdot \cos \theta} = 1.
\]

Express all terms using sine and cosine.

\[
\frac{1}{\cos^2 \theta} - 1 = \frac{\cos^2 \theta}{\cos^2 \theta}.
\]

Subtract on the left. Multiply on the right.

\[
1 - \sin^2 \theta = \cos^2 \theta.
\]

Simplify the left side.

www.algebra2.com/extra_examples

Lesson 14-4  Verifying Trigonometric Identities 783
About the Exercises...
Organization by Objective
• Transform One Side of an Equation: 11–24, 26–30
• Transform Both Sides of an Equation: 25

Odd/Even Assignments
Exercises 11–30 are structured so that students practice the same concepts whether they are assigned odd or even problems.
Alert! Exercises 37–42 require a graphing calculator.

Assignment Guide
Basic: 11–31 odd, 33–36, 43–54
Average: 11–31 odd, 33–36, 43–54 (optional: 37–42)
Advanced: 12–30 even, 31–50 (optional: 51–54)

Open-Ended Assessment
Speaking Have students explain some of the techniques they have seen or used to verify trigonometric identities in this lesson.

Getting Ready for Lesson 14-5
PREREQUISITE SKILL Students will simplify radical expressions in the process of using the Sum and Difference of Angles Formulas in Lesson 14-5. Use Exercises 51–54 to determine your students’ familiarity with simplifying radical expressions.

Assessment Options
Quiz (Lessons 14-3 and 14-4) is available on p. 893 of the Chapter 14 Resource Masters.
Mid-Chapter Test (Lessons 14-1 through 14-4) is available on p. 895 of the Chapter 14 Resource Masters.

Check for Understanding

Concept Check
1–3. See pp. 811A–811N.

1. Explain the steps used to verify the identity \( \sin \theta \tan \theta = \sec \theta - \cos \theta \).
2. Describe the various methods you can use to show that two trigonometric expressions form an identity.
3. OPEN ENDED Write a trigonometric equation that is not an identity. Explain how you know it is not an identity.

Guided Practice

GUIDED PRACTICE KEY

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Verify that each of the following is an identity. 4–9. See pp. 811A–811N.

4. \( \tan \theta (\cot \theta + \tan \theta) = \sec^2 \theta \)
5. \( \tan^2 \theta \cos^2 \theta = 1 - \cos^2 \theta \)
6. \( \frac{\cos \theta + \tan \theta}{1 - \sin \theta} = 1 + \sin \theta \)
7. \( \frac{1 + \tan^2 \theta}{\sec^2 \theta} = \tan^2 \theta \)
8. \( \sin \theta = \frac{1}{\sec \theta + \cot \theta} \)
9. \( \frac{\sec \theta + 1}{\tan \theta} = \frac{\tan \theta}{\sec \theta - 1} \)

OPEN ENDED Which expression is equivalent to \( \frac{\sec \theta + \csc \theta}{1 + \tan \theta} \)?

Standardized Test Practice

A. \( \sin \theta \)
B. \( \cos \theta \)
C. \( \tan \theta \)
D. \( \csc \theta \)

Practice and Apply

For Exercises
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</table>

Extra Practice
See page 860.

Verify that each of the following is an identity. 11–28. See pp. 811A–811N.

11. \( \cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1 \)
12. \( \cot \theta (\cot \theta + \tan \theta) = \csc^2 \theta \)
13. \( 1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta \)
14. \( \sin \theta \sec \theta \cot \theta = \theta \)
15. \( \frac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2 \)
16. \( \frac{1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta \)
17. \( \cot \theta \csc \theta = \frac{\csc \theta + \cot \theta}{\sin \theta + \tan \theta} \)
18. \( \sin \theta + \cos \theta = \frac{1 + \tan \theta}{\sec \theta} \)
19. \( \frac{\sec \theta - \sin \theta}{\sin \theta} = \cot \theta \)
20. \( \frac{\sin \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{\sin \theta} = 2 \csc \theta \)
21. \( \frac{1 + \sin \theta}{\sin \theta} = \frac{\cot \theta}{\csc \theta - 1} \)
22. \( \frac{1 + \tan \theta}{1 + \cot \theta} = \frac{\sin \theta}{\cos \theta} \)
23. \( \frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta} = 1 \)
24. \( 1 + \frac{1}{\cos \theta} = \frac{\tan^2 \theta}{\sec \theta - 1} \)
25. \( 1 - \tan^2 \theta = 2 \sec^2 \theta - \sec^4 \theta \)
26. \( \cos^2 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta \)
27. \( \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} \)
28. \( \frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \)

29. Verify that \( \tan \theta \sin \theta \cos \theta \csc^2 \theta = 1 \) is an identity. See pp. 811A–811N.
30. Show that \( 1 + \cos \theta \) and \( \frac{\sin^2 \theta}{1 - \cos \theta} \) form an identity. See pp. 811A–811N.

PHYSICS For Exercises 31 and 32, use the following information.
If an object is propelled from ground level, the maximum height that it reaches is given by \( h = \frac{v^2 \sin^2 \theta}{2g} \), where \( \theta \) is the angle between the ground and the initial path of the object, \( v \) is the object’s initial velocity, and \( g \) is the acceleration due to gravity, 9.8 meters per second squared.
31. Verify the identity \( \frac{v^2 \sin^2 \theta}{2g} = \frac{v^2 \tan^2 \theta}{2g \sec^2 \theta} \). See pp. 811A–811N.
32. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80° with the ground. Find the maximum height of the rocket. 598.7 m

Daily Intervention

Differentiated Instruction

Interpersonal Have groups or pairs of students work together to verify some of the identities in Exercises 4–9. Have students record the techniques they found helpful. Ask students to compare their list of techniques to the list of suggestions given above Example 3 on p. 783.
33. **CRITICAL THINKING** Present a logical argument for why the identity 
\[ \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \] 
is true when \( 0 \leq x \leq 1 \). See margin.

34. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See pp. 811A–811N.

How can you verify trigonometric identities?

Include the following in your answer:

- an explanation of why you cannot perform operations to each side of an unverified identity,
- an explanation of how you can tell if two expressions are equivalent, and
- an explanation of why you cannot use the graphs of two equations to verify an identity.

35. Which of the following is not equivalent to \( \cos \theta \)?

\[ \cos \frac{\theta}{\cos^2 \theta + \sin^2 \theta} \quad \text{or} \quad \frac{1 - \sin^2 \theta}{\cos \theta} \quad \text{or} \quad \cot \theta \tan \theta \quad \text{or} \quad \tan \theta \csc \theta \]

36. Which of the following is equivalent to \( \sin \theta + \cot \theta \cos \theta \)?

\[ 2 \sin \theta \quad \text{or} \quad \frac{1}{\sin \theta} \quad \cos^2 \theta \quad \text{or} \quad \frac{\sin \theta + \cos \theta}{\sin^2 \theta} \]

**Graphing Calculator**

Verifying Trigonometric Identities

You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation might be an identity. The equation has to be algebraically verified to ensure that it is an identity.

Determine whether each of the following may be or is not an identity.

37. \( \cot x + \tan x = \csc x \cot x \) is not

38. \( \sec^2 x - 1 = \sin^2 x \) may be

39. \( (1 + \sin x)(1 - \sin x) = \cos^2 x \) may be

40. \( \frac{1}{\sec x \tan x} = \csc x - \sin x \) may be

41. \( \frac{\sec^2 x}{\tan x} = \sec x \csc x \) may be

42. \( \frac{1}{\sec x} + \frac{1}{\csc x} = 1 \) is not

**Enrichment, p. 860**

**Hero’s Formula**

Hero’s formula can be used to find the area of a triangle if you know the lengths of the three sides. Consider any triangle \( \triangle ABC \). Let \( a \), \( b \), and \( c \) represent the length of sides \( \triangle ABC \). Then

\[ \text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \]

where \( s \) is the semiperimeter; that is, \( s = \frac{a+b+c}{2} \). The formula determines the area of the triangle. Draw a line perpendicular to one of the sides of the triangle to determine the height of the triangle. The area is the base times the height divided by two. Use the Pythagorean Theorem to determine the height of the triangle. Combine this idea with the formula to calculate the area of a triangle. Help students understand the formula by changing each side of the triangle. Then carefully explain the area of the triangle as the product of the base and height divided by two.

Helping You Remember

- Many students have trouble knowing where to start in verifying trigonometric identities. What is a simple technique you can recommend that you can always use if you don’t see a quicker approach? Sample answer: Write both sides in terms of sine and cosine. Then simplify each side as much as possible.
Focus

5-Minute Check Transparency 14-5 Use as a quiz or review of Lesson 14-4.

Mathematical Background notes are available for this lesson on p. 760D.

How are the sum and difference formulas used to describe communication interference?

Ask students:
• Where else have you heard the term interference? Sample answer: television and radio
• At its peak, how does the amplitude of the combined wave compare to the amplitude of the initial two waves? The amplitude of the combined wave is the sum of the amplitudes of the two initial waves.
• Why does the combined wave cross the x-axis at a point where neither of the two initial waves are crossing the axis? The combined wave is the sum of the other two waves. It crosses the x-axis at points where one of the initial waves is above the x-axis and the other wave is an equal distance below the x-axis.

Study Tips

Reading Math
The Greek letter beta, β, can be used to denote the measure of an angle.

It is important to realize that sin (α + β) is not the same as sin α ± sin β.

SUM AND DIFFERENCE FORMULAS Notice that the third equation shown above involves the sum of α and β. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find sin 15° by evaluating sin (60° - 45°). Formulas can be developed that can be used to evaluate expressions like sin (α - β) or cos (α + β).

The figure at the right shows two angles α and β in standard position on the unit circle. Use the Distance Formula to find d, where (x₁, y₁) = (cos β, sin β) and (x₂, y₂) = (cos α, sin α).

\[
\begin{align*}
d &= \sqrt{(\cos α - \cos β)^2 + (\sin α - \sin β)^2} \\
d^2 &= (\cos α - \cos β)^2 + (\sin α - \sin β)^2 \\
d^2 &= (\cos^2 α - 2\cos α \cos β + \cos^2 β) + (\sin^2 α - 2\sin α \sin β + \sin^2 β) \\
d^2 &= \cos^2 α + \sin^2 α + \cos^2 β + \sin^2 β - 2 \cos α \cos β - 2 \sin α \sin β \\
d^2 &= 1 + 1 - 2 \cos α \cos β - 2 \sin α \sin β \\
n &\text{ and } \\
d^2 &= 2 - 2 \cos α \cos β - 2 \sin α \sin β \\
&\text{ or } \sin^2 β + \cos^2 β = 1
\end{align*}
\]

Now find the value of d² when the angle having measure α - β is in standard position on the unit circle, as shown in the figure at the left.

\[
\begin{align*}
d &= \sqrt{[\cos (α - β) - 1]^2 + [\sin (α - β) - 0]^2} \\
d^2 &= [\cos (α - β) - 1]^2 + [\sin (α - β) - 0]^2 \\
n &= [\cos^2 (α - β) - 2 \cos (α - β) + 1] + \sin^2 (α - β) \\
&= \cos^2 (α - β) + \sin^2 (α - β) - 2 \cos (α - β) + 1 \\
&= 1 - 2 \cos (α - β) + 1 \\
&= 2 - 2 \cos (α - β)
\end{align*}
\]

What You’ll Learn
• Find values of sine and cosine involving sum and difference formulas.
• Verify identities by using sum and difference formulas.

How are the sum and difference formulas used to describe communication interference?

Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. Constructive interference occurs when two waves combine to have a greater amplitude than either of the component waves. Destructive interference occurs when the component waves combine to have a smaller amplitude.
By equating the two expressions for \( d^2 \), you can find a formula for \( \cos (\alpha - \beta) \).

\[
d^2 = a^2 - 2a \cdot c \cos (\alpha - \beta)
\]

\[
2 - 2 \cos (\alpha - \beta) = 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta
\]

\[-1 + \cos (\alpha - \beta) = -1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad \text{Divide each side by } -2.
\]

\[
cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad \text{Add 1 to each side.}
\]

Use the formula for \( \cos (\alpha - \beta) \) to find a formula for \( \cos (\alpha + \beta) \).

\[
\cos (\alpha - \beta) = \cos [\alpha - (-\beta)]
\]

\[
= \cos \alpha \cos (-\beta) + \sin \alpha \sin (-\beta)
\]

\[
= \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\cos \beta = \cos \beta \quad \text{and} \quad \sin (-\beta) = -\sin \beta
\]

You can use a similar method to find formulas for \( \sin (\alpha + \beta) \) and \( \sin (\alpha - \beta) \).

**Key Concept**

**Sum and Difference of Angles Formulas**

The following identities hold true for all values of \( \alpha \) and \( \beta \).

\[
\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
\]

\[
\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta
\]

Notice the symbol \( \mp \) in the formula for \( \cos (\alpha \pm \beta) \). It means “minus or plus.” In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

**Example 1**

Use Sum and Difference of Angles Formulas

Find the exact value of each expression.

a. \( \cos 75^\circ \)

Use the formula \( \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \).

\[
\cos 75^\circ = \cos (30^\circ + 45^\circ)
\]

\[
= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ
\]

\[
= \left( \frac{\sqrt{3}}{2} \right) \left( \frac{\sqrt{2}}{2} \right) - \left( \frac{1}{2} \right) \left( \frac{\sqrt{2}}{2} \right)
\]

Evaluate each expression.

\[
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

Multiply.

\[
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

Simplify.

b. \( \sin (-210^\circ) \)

Use the formula \( \sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \).

\[
\sin (-210^\circ) = \sin (60^\circ - 270^\circ)
\]

\[
= \sin 60^\circ \cos 270^\circ - \cos 60^\circ \sin 270^\circ
\]

\[
= \left( \frac{\sqrt{3}}{2} \right)(0) - \left( \frac{1}{2} \right)(-1)
\]

Evaluate each expression.

\[
= 0 - \left( -\frac{1}{2} \right)
\]

Multiply.

\[
= \frac{1}{2}
\]

Simplify.

www.algebra2.com/extra_examples

Lesson 14-5   Sum and Difference of Angles Formulas   787

**Differentiated Instruction**

**Naturalist** Have students apply what they know about geography, or have them conduct research, to find out whether the light energy per square foot would be increasing or decreasing as you travel toward the equator. Have students research the latitude of your city, or a city that interests them, and repeat Example 2 for that city.
Verify that each of the following is an identity.

a. \( \cos (90^\circ - \theta) = \sin \theta \)
\[
\cos (90^\circ - \theta) = \sin \theta
\]

b. \( \cos (180^\circ - \theta) = -\cos \theta \)
\[
\cos (180^\circ - \theta) = -\cos \theta
\]

### Example 2
Use Sum and Difference Formulas to Solve a Problem

**PHYSICS** On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by

\[
E \sin (113.5^\circ - \phi)
\]

where \( \phi \) is the latitude of the location and \( E \) is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.

Use the difference formula for sine.

\[
sin (113.5^\circ - \phi) = sin 113.5^\circ \cos \phi - \cos 113.5^\circ \sin \phi
\]

\[
= sin 113.5^\circ \cos 43.1^\circ - \cos 113.5^\circ \sin 43.1^\circ
\]

\[
= 0.9171 \cdot 0.7301 - (-0.3987) \cdot 0.6833
\]

\[
= 0.9420
\]

In Rochester, New York, the maximum light energy per square foot is 0.9420E.

### Example 3
Verify Identities

Verify that each of the following is an identity.

a. \( \sin (180^\circ + \theta) = -\sin \theta \)
\[
\sin (180^\circ + \theta) = -\sin \theta \quad \text{Original equation}
\]

b. \( \cos (180^\circ + \theta) = -\cos \theta \)
\[
\cos (180^\circ + \theta) = -\cos \theta \quad \text{Original equation}
\]

### Check for Understanding

1. Determine whether \( \sin (\alpha + \beta) = \sin \alpha + \sin \beta \) is an identity.
2. Describe a method for finding the exact value of \( \sin 105^\circ \). Then find the value.
3. OPEN ENDED Determine whether \( \cos (\alpha - \beta) < 1 \) is sometimes, always, or never true. Explain your reasoning.

#### Guided Practice

Find the exact value of each expression.

4. \( \sin 75^\circ = \sqrt{6} + \sqrt{2} \)
5. \( \sin 165^\circ = \frac{\sqrt{6} - \sqrt{2}}{4} \)
6. \( \cos 255^\circ = \frac{\sqrt{2} - \sqrt{6}}{4} \)
7. \( \cos (-30^\circ) = \frac{\sqrt{3}}{2} \)
8. \( \sin (-240^\circ) = \frac{\sqrt{3}}{2} \)
9. \( \cos (-120^\circ) = -\frac{1}{2} \)

Verify that each of the following is an identity.

10. \( \cos (270^\circ - \theta) = -\sin \theta \)
11. \( \sin \left( \theta + \frac{\pi}{2} \right) = \cos \theta \)
12. \( \sin (\theta + 30^\circ) + \cos (\theta + 60^\circ) = \cos \theta \)

Answers

1. \( \sin (\alpha + \beta) \neq \sin \alpha + \sin \beta \)
\[
\sin \alpha \cos \beta + \cos \alpha \sin \beta \neq \sin \alpha + \sin \beta
\]

2. Use the formula \( \sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \).

Since \( \sin 105^\circ = \sin (60^\circ + 45^\circ) \), replace \( \alpha \) with \( 60^\circ \) and \( \beta \) with \( 45^\circ \) to get \( \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \). By finding the sum of the products of the values, the result is \( \frac{\sqrt{6} + \sqrt{2}}{4} \) or about 0.9659.
Homework Help
For Exercises Use Examples
14-27 1
28-39 3
40-41 2
42-43 4

Extra Practice
See page 860.

17. \(-\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\)
18. \(-\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\)
19. \(-\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\)
23. \(-\frac{2}{3}\)
24. \(-\frac{2}{3}\)
25. \(-\frac{2}{3}\)

More About...
Physics
In the northern hemisphere, the day with the least number of hours of daylight is December 21 or 22, the first day of winter.
Source: www.timeplease.com

Practice and Apply

Find the exact value of each expression.

14. \(\sin 135\degree \cdot \frac{\sqrt{2}}{2}\)
15. \(\cos 105\degree \frac{\sqrt{2}-\sqrt{6}}{4}\)
16. \(\sin 285\degree -\frac{\sqrt{6}+\sqrt{2}}{4}\)
17. \(\cos 165\degree -\frac{\sqrt{2}}{4}\)
18. \(\cos 195\degree -\frac{\sqrt{2}}{4}\)
19. \(\sin 255\degree -\frac{\sqrt{6}+\sqrt{2}}{4}\)
20. \(\sin 225\degree -\frac{\sqrt{2}}{4}\)
21. \(\sin 315\degree -\frac{\sqrt{2}}{4}\)
22. \(\sin (-15\degree) -\frac{\sqrt{6}-\sqrt{2}}{4}\)
23. \(\cos (-45\degree)\)
24. \(\cos (-150\degree)\)

26. What is the exact value of \(\tan 75\degree - \sin 15\degree\)\?
27. Find the exact value of \(\cos 105\degree + \cos 225\degree\). \(-\frac{\sqrt{6}+\sqrt{2}}{4}\)

Verify that each of the following is an identity. 28-39. See pp. 811A-811N.

28. \(\sin (270\degree - \theta) = -\cos \theta\)
29. \(\cos (90\degree + \theta) = -\sin \theta\)
30. \(\cos (90\degree - \theta) = \sin \theta\)
31. \(\sin (90\degree - \theta) = \cos \theta\)
32. \(\sin \left(\theta + \frac{3\pi}{2}\right) = -\cos \theta\)
33. \(\cos \left(\pi - \theta\right) = -\cos \theta\)
34. \(\cos (2\pi + \theta) = \cos \theta\)
35. \(\sin (\pi - \theta) = \sin \theta\)
36. \(\sin (60\degree + \theta) + \sin (60\degree - \theta) = \sqrt{3} \cos \theta\)
37. \(\sin \left(\theta + \frac{\pi}{3}\right) - \cos \left(\theta + \frac{\pi}{6}\right) = \sin \theta\)

PHYSICS For Exercises 42-45, use the following information.

On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by \(E = \sin(113.5\degree + \phi)\), where \(\phi\) is the latitude of the location. Use the sum of angles formula to find the amount of light energy, in terms of \(E\), for each location. \(42. 0.3681 \, E \quad 43. 0.4179 \, E\)

44. Charleston, SC (Latitude: 32.7° N)
45. San Diego, CA (Latitude: 32.7° N)

0.6157 \, E \quad 0.5563 \, E

46. CRITICAL THINKING
Use the sum and difference formulas for sine and cosine to derive formulas for \(\tan(\alpha + \beta)\) and \(\tan(\alpha - \beta)\). See pp. 811A-811N.

Enrichment, p. 866

Identities for the Products of Sines and Cosines

By adding the identities for the sums of the sine and difference of the cosine of two angles, a non-identity is obtained.

\[
\sin \alpha \cos \beta = \frac{1}{2} \left(\sin(\alpha + \beta) + \sin(\alpha - \beta)\right)
\]

This new identity is useful for expressing certain products as sums.

Example
Write \(\sin 3\theta \cos \alpha \) as a sum.

Consider the identity \(\cos \theta = \cos(\theta + 0) = \cos(\theta + 0) + \cos(\theta - 0)\). Thus, \(\sin 3\theta \cos \alpha = \frac{1}{2} \left(\sin(3\theta + \alpha) + \sin(3\theta - \alpha)\right)\).

Helping You Remember
1. Some students have trouble remembering which signs to use on the right-hand sides of the sum and difference of angles formulas. What is one way to remember this?
2. Sometimes, the sign on the right side is the same as the one on both sides. In these identities, the signs are opposite on the two sides.

Lesson 14-5 Sum and Difference of Angles Formulas 789

Study Guide and Intervention, p. 861 (shown) and p. 862

Sum and Difference Formulas: The following formulas are used for evaluating an expression like \(\sin(120\degree)\) from the known values of \(\sin 90\degree\) and \(\sin 60\degree\).

\[
\begin{align*}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta
\end{align*}
\]

Skills Practice, p. 863 and Practice, p. 864 (shown)

Find the exact value of each expression.

1. \(\sin 45\degree \cdot \frac{\sqrt{2}}{2}\)
2. \(\cos 30\degree \cdot \frac{\sqrt{3}}{2}\)
3. \(\cos 60\degree \cdot \frac{1}{2}\)
4. \(\tan 30\degree \cdot \frac{1}{\sqrt{3}}\)
5. \(\sec 45\degree \cdot \sqrt{2}\)
6. \(\csc 45\degree \cdot \sqrt{2}\)

In Exercises 15 and 16, use the following information.

15. San Diego, CA (Latitude: 32.7° N)
16. Los Angeles, CA (Latitude: 33.9° N)

ELECTRICITY For Exercises 15 and 16, see the following information.

In a certain circuit carrying alternating current, the formula \(I = 2 \sin(100t + 180\degree)\) can be used to find the current, in amperes, that passes through the circuit at time \(t\) seconds. \(t = 0\) represents the instant when the current is at its maximum value.

15. Determine the current at \(t = 1\) second. \(2 \sin 180\degree\)

16. Find the current after 3 seconds. \(2 \sin(100 \cdot 3 + 180\degree)\)

Pre-Activity How are the sine and cosine formulas used to describe communication interference?

Read the introduction to Lesson 14-5 on page 786 in your textbook.

Consider the functions \(y = \sin x\) and \(y = \cos x\). In the graph of these two functions, how constructive interference or destructive interference constructive?

Reading the Lesson
1. Match each expression from the list on the left with an expression from the list on the right that is equal to it for all values of the variables. (Some of the expressions from the list on the right may be used more than once or not at all.)

To find the current at \(t = 1\) second, use \(2 \sin 180\degree\) amperes. 

To find the current after 3 seconds, use \(2 \sin(100 \cdot 3 + 180\degree)\) amperes.

Helping You Remember

2. Make expressions equal to \(\sin 120\degree\). (There may be more than one correct choice.)
A. \(\sin 75\degree \cdot \cos 45\degree \cdot \cos 30\degree\) B. \(\sin 15\degree \cdot \cos 60\degree \cdot \sin 45\degree\) C. \(\sin 60\degree \cdot \cos 45\degree \cdot \sin 45\degree\) D. \(\cos 60\degree \cdot \cos 45\degree \cdot \sin 45\degree\)
Open-Ended Assessment

Modeling  Show students a graph of the function \( y = \sin x \). Have students point out on the graph which values would be most useful to use with the sum and difference of angles formulas, and ask them to explain their reasoning.

Getting Ready for Lesson 14-6

PREREQUISITE SKILL Students will find values using half-angle formulas in Lesson 14-6. The half-angle formulas include expressions within square root symbols, so students must be comfortable evaluating square roots. Use Exercises 67–74 to determine your students’ familiarity with the Square Root Property.

Answers

47. Sample answer: To determine communication interference, you need to determine the sine or cosine of the sum or difference of two angles. Answers should include the following information.

• Interference occurs when waves pass through the same space at the same time. When the combined waves have a greater amplitude, constructive interference results and when the combined waves have a smaller amplitude, destructive interference results.

48. Find the exact value of \( \sin \theta \).  
   \[
   \frac{\sqrt{3}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{3}
   \]

49. Find the exact value of \( \cos (-210^\circ) \).  
   \[
   \frac{\sqrt{3}}{2} \quad 0.5 \quad -\frac{\sqrt{3}}{2} \quad -0.5
   \]

Maintain Your Skills

Mixed Review

Verify that each of the following is an identity. (Lesson 14-4)

50. \( \cot \theta + \sec \theta = \frac{\cos^2 \theta + \sin \theta}{\sin \theta \cos \theta} \)

51. \( \sin^2 \theta + \tan^2 \theta = (1 - \cos^2 \theta) + \frac{\sec^2 \theta}{\csc^2 \theta} \)

52. \( \sin \theta (\sin \theta + \csc \theta) = 2 - \cos^2 \theta \)

53. \( \frac{\sec \theta}{\tan \theta} = \csc \theta \)

Simplify each expression. (Lesson 14-3)

54. \( \frac{\tan \theta \csc \theta}{\sec \theta} = 1 \)

55. \( 4\left(\sec^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta}\right) \)

56. \( (\cot \theta + \tan \theta)\sin \theta \sec \theta \)

Find the exact values of the six trigonometric functions of \( \theta \) if the terminal side of \( \theta \) in standard position contains the given point. (Lesson 13-3) 58–60. See margin

58. \( (5, -3) \)

59. \( (-3, -4) \)

60. \( (0, 2) \)

Evaluate each expression. (Lesson 12-2)

61. \( P(6, 4) \)

62. \( P(12, 7) \)

63. \( C(8, 3) \)

64. \( C(10, 4) \)

65. AVIATION A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? (Lesson 13-5)

66. Write \( 6y^2 - 34x^2 = 204 \) in standard form. (Lesson 8-5)

67. \( x^2 = \frac{20}{16} \) \( \pm \frac{2\sqrt{5}}{2} \)

68. \( x^2 = \frac{9}{25} \) \( \pm \frac{3}{5} \)

69. \( x^2 = \frac{5}{25} \) \( = \frac{\sqrt{6}}{5} \)

70. \( x^2 = 18 \) \( \pm \frac{3}{4} \)

71. \( x^2 - 1 = \frac{1}{2} \) \( = \frac{\sqrt{6}}{2} \)

72. \( x^2 - 1 = \frac{4}{5} \)

73. \( x^2 = \frac{\sqrt{6}}{2} \) \( = 1 \)

74. \( x^2 = \frac{\sqrt{6}}{2} \) \( = 1 \)

PRE不见得 SKILL  Solve each equation.

(To review solving equations using the Square Root Property, see Lesson 6-4.)

72. \( \pm \frac{3\sqrt{5}}{5} \)

76. Write \( 6y^2 - 34x^2 = 204 \) in standard form. (Lesson 8-5)

77. \( x^2 = \frac{20}{16} \) \( \pm \frac{2\sqrt{5}}{2} \)

78. \( x^2 = \frac{9}{25} \) \( \pm \frac{3}{5} \)

79. \( x^2 = \frac{5}{25} \) \( = \frac{\sqrt{6}}{5} \)

71. \( x^2 - 1 = \frac{1}{2} \) \( = \frac{\sqrt{6}}{2} \)

72. \( x^2 - 1 = \frac{4}{5} \)

73. \( x^2 = \frac{\sqrt{6}}{2} \) \( = 1 \)

74. \( x^2 = \frac{\sqrt{6}}{2} \) \( = 1 \)

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Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called nodes, that do not move under the vibration. Halfway between each pair of consecutive nodes are antinodes that undergo the maximum vibration. The nodes and antinodes form harmonics. These harmonics can be represented using variations of the equations $y = \sin 2\theta$ and $y = \sin \frac{1}{2} \theta$.

**Double-Angle Formulas** You can use the formula for $\sin (\alpha + \beta)$ to find the sine of twice an angle $\theta$, $\sin 2\theta$, and the formula for $\cos (\alpha + \beta)$ to find the cosine of twice an angle $\theta$, $\cos 2\theta$.

\[
\begin{align*}
\sin 2\theta &= \sin (\theta + \theta) \\
&= \sin \theta \cos \theta + \cos \theta \sin \theta \\
&= 2 \sin \theta \cos \theta \\
\cos 2\theta &= \cos (\theta + \theta) \\
&= \cos \theta \cos \theta - \sin \theta \sin \theta \\
&= \cos^2 \theta - \sin^2 \theta \\
&= 1 - 2 \sin^2 \theta
\end{align*}
\]

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2 \theta - \sin^2 \theta$.

\[
\begin{align*}
\cos^2 \theta - \sin^2 \theta &= (1 - \sin^2 \theta) - \sin^2 \theta \\
&= 1 - 2 \sin^2 \theta \\
&= 2 \cos^2 \theta - 1
\end{align*}
\]

These formulas are called the double-angle formulas.

**Key Concept**

Double-Angle Formulas

The following identities hold true for all values of $\theta$.

\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cos \theta \\
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= 1 - 2 \sin^2 \theta \\
&= 2 \cos^2 \theta - 1
\end{align*}
\]
Teaching Tip  Remind students of the identity \(\sin^2 \theta + \cos^2 \theta = 1\), and its two variations that occur by subtracting either \(\sin^2 \theta\) or \(\cos^2 \theta\) from both sides. Explain that there are three versions of the formula for \(\cos 2\theta\) because of the three variations of the identity \(\sin^2 \theta + \cos^2 \theta = 1\).

In-Class Example

Find the exact value of each expression if \(\sin \theta = \frac{3}{4}\) and \(\theta\) is between 0° and 90°.

1. \(\sin 2\theta\)
2. \(\cos 2\theta\)

\[\sin 2\theta = 2 \sin \theta \cos \theta\]
\[= 2 \left(\frac{3}{4}\right) \left(\frac{\sqrt{7}}{4}\right) = \frac{3\sqrt{7}}{8}\]

\[\cos 2\theta = 1 - 2 \sin^2 \theta\]
\[= 1 - 2 \left(\frac{3}{4}\right)^2 = 1 - 2 \left(\frac{9}{16}\right) = \frac{1}{8}\]

Example 1 Double-Angle Formulas

Find the exact value of each expression if \(\sin \theta = \frac{4}{5}\) and \(\theta\) is between 90° and 180°.

a. \(\sin 2\theta\)

Use the identity \(\sin 2\theta = 2 \sin \theta \cos \theta\).
First, find the value of \(\cos \theta\).
\[\cos^2 \theta = 1 - \sin^2 \theta\]
\[= 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}\]
\[\cos \theta = \pm \frac{3}{5}\]
\[\text{Subtract.}\]
Find the square root of each side.

Since \(\theta\) is in the second quadrant, cosine is negative. Thus, \(\cos \theta = -\frac{3}{5}\).

Now find \(\sin 2\theta\).
\[\sin 2\theta = 2 \sin \theta \cos \theta\]
\[= 2 \left(\frac{4}{5}\right) \left(-\frac{3}{5}\right) = -\frac{24}{25}\]

b. \(\cos 2\theta\)

Use the identity \(\cos 2\theta = 1 - 2 \sin^2 \theta\).
\[\cos 2\theta = 1 - 2 \sin^2 \theta\]
\[= 1 - 2 \left(\frac{4}{5}\right)^2 = 1 - 2 \left(\frac{16}{25}\right) = \frac{7}{25}\]

HALF-ANGLE FORMULAS You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find \(\sin \frac{\alpha}{2}\):

\[1 - 2 \sin^2 \theta = \cos 2\theta\]  
\[\text{Double-angle formula}\]
\[1 - 2 \sin^2 \frac{\alpha}{2} = \cos \alpha\]  
Substitute \(\alpha\) for \(\theta\) and \(\alpha\) for \(2\theta\).
\[\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}\]  
Solve for \(\sin^2 \frac{\alpha}{2}\).
\[\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}\]  
Take the square root of each side.

Find \(\cos \frac{\alpha}{2}\):

\[2 \cos^2 \theta - 1 = \cos 2\theta\]  
\[\text{Double-angle formula}\]
\[2 \cos^2 \frac{\alpha}{2} - 1 = \cos \alpha\]  
Substitute \(\alpha\) for \(\theta\) and \(\alpha\) for \(2\theta\).
\[\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}\]  
Solve for \(\cos^2 \frac{\alpha}{2}\).
\[\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}\]  
Take the square root of each side.
These are called the half-angle formulas. The signs are determined by the function of \( \frac{\alpha}{2} \).

### Key Concept

**Half-Angle Formulas**

The following identities hold true for all values of \( \alpha \):

\[
\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\
\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}
\]

### Example 2

**Half-Angle Formulas**

Find \( \cos \frac{\alpha}{2} \) if \( \sin \alpha = -\frac{3}{4} \) and \( \alpha \) is in the third quadrant.

Since \( \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \), we must find \( \cos \alpha \) first.

\[
\cos^2 \alpha = 1 - \sin^2 \alpha \\
\cos^2 \alpha = 1 - \left(-\frac{3}{4}\right)^2 \\
\cos^2 \alpha = 1 - \frac{9}{16} \\
\cos^2 \alpha = \frac{7}{16}
\]

Simplify.

\[\cos \alpha = \pm \frac{\sqrt{7}}{4}\]

Take the square root of each side.

Since \( \alpha \) is in the third quadrant, \( \cos \alpha = -\frac{\sqrt{7}}{4} \).

\[
\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{(Half-angle formula)}
\]

\[
= \pm \sqrt{\frac{1 - \frac{\sqrt{7}}{4}}{2}} \\
= \pm \sqrt{\frac{4 - \sqrt{7}}{8}} \quad \text{Simplify the radicand.}
\]

\[
= \pm \frac{\sqrt{4 - \sqrt{7}}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \quad \text{Rationalize.}
\]

\[
= \pm \frac{\sqrt{8 - 2\sqrt{7}}}{4} \quad \text{Multiply.}
\]

Since \( \alpha \) is between 180° and 270°, \( \frac{\alpha}{2} \) is between 90° and 135°. Thus, \( \cos \frac{\alpha}{2} \) is negative and equals \( -\frac{\sqrt{8 - 2\sqrt{7}}}{4} \).

### Example 3

**Evaluate Using Half-Angle Formulas**

Find the exact value of each expression by using the half-angle formulas.

a. \( \sin 105^\circ \)

\[
\sin 105^\circ = \sin \frac{210^\circ}{2} \\
= \sqrt{\frac{1 - \cos 210^\circ}{2}} \\
= \sqrt{\frac{1 - (-\sqrt{3}/2)}{2}} \\
= \sqrt{\frac{1 + \sqrt{3}/2}{2}}
\]

(continued on the next page)

www.algebra2.com/extra_examples

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### DAILY INTERVENTION

**Auditory/Musical** If possible, ask a music teacher at your school to talk to students about harmonics. Students playing stringed instruments may also be willing to share what they have learned about harmonics and waves. If a music teacher is not available, a physics teacher may also be able to demonstrate harmonics or bring a device that creates standing waves to class.
**In-Class Examples**

3. Find the exact value of each expression by using the half-angle formulas.
   a. \( \sin 165^\circ \)
   b. \( \cos \frac{9\pi}{8} \)

4. Verify that \( \sin \theta (\cos^2 \theta - \cos 2\theta) = \sin^3 \theta \) is an identity.

\[
\begin{align*}
\sin (\cos^2 \theta - \cos 2\theta) & \quad \cos \theta \cos \theta - \cos 2\theta \\
& \quad \cos \theta \cos \theta - (\cos \theta \sin \theta - \sin^2 \theta) \\
& \quad \cos \theta \cos \theta - \sin \theta (\sin \theta + \sin^2 \theta) \\
& \quad \sin \theta (\sin^2 \theta) \\
& \quad \sin^3 \theta
\end{align*}
\]

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

**Example 4** Verify Identities

Verify that \( (\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta \) is an identity.

\[
\begin{align*}
& (\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta \\
& \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta = 1 + \sin 2\theta \\
& 1 + 2 \sin \theta \cos \theta = 1 + \sin 2\theta \\
& \sin^2 \theta + \cos^2 \theta = 1 \\
& 1 + \sin 2\theta = 1 + \sin 2\theta \quad \text{Double-angle formula}
\end{align*}
\]

**Check for Understanding**

**Concept Check**

1. Explain how to find \( \cos \frac{x}{2} \) if \( x \) is in the third quadrant.
2. Find a counterexample to show that \( \cos 2\theta = 2 \cos \theta \) is not an identity.
3. OPEN ENDED Describe the conditions under which you would use each of the three identities for \( \cos 2\theta \).

**Guided Practice**

Find the exact values of \( \sin 2\theta \), \( \cos 2\theta \), and \( \sin \frac{\theta}{2} \) and \( \cos \frac{\theta}{2} \) for each of the following.

4. \( \cos \theta = \frac{3}{5}, 90^\circ < \theta < 90^\circ \)
5. \( \cos \theta = -\frac{2}{3}, 180^\circ < \theta < 270^\circ \)
6. \( \sin \theta = \frac{1}{2}, 0^\circ < \theta < 90^\circ \)
7. \( \sin \theta = -\frac{3}{4}, 270^\circ < \theta < 360^\circ \)

Find the exact value of each expression by using the half-angle formulas.

8. \( \sin 195^\circ = \frac{-\sqrt{2 - \sqrt{3}}}{2} \)
9. \( \cos \frac{19\pi}{12} = \frac{\sqrt{2 - \sqrt{3}}}{2} \)

Answers

1. Sample answer: If \( x \) is in the third quadrant, then \( \frac{x}{2} \) is between 90° and 135°. Use the half-angle formula for cosine knowing that the value is negative.
2. Sample answer: 45°; \( \cos 2(45^\circ) = \cos 90^\circ \) or 0, \( \cos 45^\circ = 2 \cdot \frac{\sqrt{2}}{2} \) or \( \sqrt{2} \)
3. Sample answer: The identity used for \( \cos 2\theta \) depends on whether you know the value of \( \sin \theta \), \( \cos \theta \), or both values.
4. \( \frac{24}{25}, \frac{7}{25}, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \)

**Guided Practice Key**

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794 Chapter 14 Trigonometric Graphs and Identities
Verify that each of the following is an identity. 10–11. See margin.
10. \[ \cot x = \frac{\sin 2x}{1 - \cos 2x} \]
11. \[ \cos^2 2x + 4 \sin^2 x \cos^2 x = 1 \]

Application
12. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If \( \theta \) is the measure of the angle at the vertex of the cone, then the Mach number \( M \) can be determined using the formula \( \sin \frac{\theta}{2} = \frac{1}{M} \). Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of 75°. 1.64

Practice and Apply

Find the exact values of \( \sin 2\theta \), \( \cos 2\theta \), \( \sin \frac{\theta}{2} \), and \( \cos \frac{\theta}{2} \) for each of the following.
13. \( \sin \theta = \frac{5}{13} \); \( 90^\circ < \theta < 180^\circ \)
14. \( \cos \theta = \frac{1}{5} \); \( 270^\circ < \theta < 360^\circ \)
15. \( \cos \theta = -\frac{1}{3} \); \( 180^\circ < \theta < 270^\circ \)
16. \( \sin \theta = -\frac{3}{5} \); \( 180^\circ < \theta < 270^\circ \)
17. \( \sin \theta = -\frac{3}{8} \); \( 270^\circ < \theta < 360^\circ \)
18. \( \cos \theta = -\frac{1}{4} \); \( 90^\circ < \theta < 180^\circ \)
19. \( \cos \theta = \frac{1}{6} \); \( 0^\circ < \theta < 90^\circ \)
20. \( \cos \theta = -\frac{12}{13} \); \( 180^\circ < \theta < 270^\circ \)
21. \( \sin \theta = -\frac{1}{3} \); \( 270^\circ < \theta < 360^\circ \)
22. \( \sin \theta = -\frac{1}{4} \); \( 180^\circ < \theta < 270^\circ \)
23. \( \cos \theta = \frac{2}{3} \); \( 0^\circ < \theta < 90^\circ \)
24. \( \sin \theta = \frac{2}{5} \); \( 90^\circ < \theta < 180^\circ \)

Find the exact value of each expression by using the half-angle formulas.
25. \( \cos 165^\circ = \frac{-\sqrt{2} + \sqrt{3}}{4} \)
26. \( \sin 225^\circ = \pm \frac{\sqrt{2} - \sqrt{3}}{2} \)
27. \( \cos 157\frac{1}{2}^\circ = \frac{-\sqrt{2} + \sqrt{3}}{4} \)
28. \( \sin 345^\circ = \frac{-\sqrt{2} - \sqrt{3}}{2} \)
29. \( \sin \frac{7\pi}{8} = \frac{\sqrt{2} - \sqrt{3}}{2} \)
30. \( \cos \frac{7\pi}{12} = \frac{\sqrt{2} + \sqrt{3}}{4} \)

Verify that each of the following is an identity. 31–36. See pp. 811A–811N.
31. \( \sin 2x = 2 \cot x \sin^2 x \)
32. \( \cos^2 x = \frac{1}{2} + \cos x \)
33. \( \sin^4 x - \cos^4 x = 2 \sin^2 x - 1 \)
34. \( \sin^2 x = \frac{1}{2}(1 - \cos 2x) \)
35. \( \tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x} \)
36. \( \frac{1}{\sin x \cos x} = \frac{\cos x}{\sin x} = \tan x \)

More About...

Optics
A rainbow appears when the sun shines through water droplets that act as prisms.

Answers
10. \( \cot x = \frac{\sin 2x}{1 - \cos 2x} \)
11. \( \cos^2 2x + 4 \sin^2 x \cos^2 x = 1 \)
12. \( \cos^2 2x + \sin^2 2x = 1 \)
13. \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \)
14. \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \)
15. \( \frac{4\sqrt{2}}{9}, \frac{7}{9}, \frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3} \)
16. \( \frac{24}{25}, \frac{7}{25}, \frac{3\sqrt{10}}{10}, \frac{-\sqrt{10}}{10} \)
17. \( \frac{-3\sqrt{55}}{32}, \frac{23}{32}, \frac{\sqrt{8 - \sqrt{55}}}{4}, \frac{-\sqrt{8 + \sqrt{55}}}{4} \)
18. \( \frac{\sqrt{15}}{8}, \frac{7}{8}, \frac{\sqrt{6}}{4}, \frac{4}{4} \)
19. \( \frac{\sqrt{35}}{18}, \frac{17}{18}, \frac{\sqrt{21}}{6} \)
20. \( \frac{120}{169}, \frac{119}{169}, \frac{5\sqrt{26}}{26}, \frac{-\sqrt{26}}{26} \)
21. \( \frac{4\sqrt{2}}{9}, \frac{7}{9}, \frac{18 - 12\sqrt{2}}{6}, \frac{-\sqrt{18 - 12\sqrt{2}}}{6} \)
22. \( \frac{\sqrt{15}}{8}, \frac{5 + 2\sqrt{15}}{8}, \frac{-\sqrt{8 - 2\sqrt{15}}}{4} \)
23. \( \frac{4\sqrt{5}}{9}, \frac{1}{9}, \frac{\sqrt{6}}{6}, \frac{\sqrt{30}}{6} \)
24. \( \frac{-4\sqrt{21}}{17}, \frac{5\sqrt{2} + 10\sqrt{21}}{5}, \frac{-\sqrt{5\sqrt{10} - 10\sqrt{21}}}{10} \)
### Double-Angle Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin 2\theta = 2\sin \theta \cos \theta )</td>
<td>( \sin 2\theta = \frac{2\tan \theta}{1 + \tan^2 \theta} )</td>
</tr>
<tr>
<td>( \cos 2\theta = \cos^2 \theta - \sin^2 \theta )</td>
<td>( \cos^2 \theta = \frac{1 + \cos 2\theta}{2} )</td>
</tr>
<tr>
<td>( \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} )</td>
<td>( \tan 2\theta = \frac{\tan \theta}{1 - \tan^2 \theta} )</td>
</tr>
</tbody>
</table>

### Example

**Find the exact values of** \( \sin 2\theta \) **and** \( \cos 2\theta \) **if** \( \sin \theta = \frac{3}{5} \) **and** \( \cos \theta = \frac{4}{5} \).

- **Solution:**
  
  \[ \sin 2\theta = 2\sin \theta \cos \theta = 2 \left( \frac{3}{5} \right) \left( \frac{4}{5} \right) = \frac{24}{25} \]
  
  \[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left( \frac{4}{5} \right)^2 - \left( \frac{3}{5} \right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25} \]

### Reading to Learn, Mathematics, p. 871

**Skills Practice, p. 869 and Practice, p. 870 (shown)**

**Find the exact values of** \( \sin 2\theta \) **and** \( \cos 2\theta \) **for each of the following**:

\[ \sin \theta = -\frac{3}{5}, \cos \theta = -\frac{4}{5} \]

- **Solution:**
  
  \[ \sin 2\theta = 2\sin \theta \cos \theta = 2 \left( -\frac{3}{5} \right) \left( -\frac{4}{5} \right) = \frac{24}{25} \]
  
  \[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left( -\frac{4}{5} \right)^2 - \left( -\frac{3}{5} \right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25} \]

### Critical Thinking

**Find the exact values of** \( \sin 2\theta \) **and** \( \cos 2\theta \) **for each of the following**:

\[ \sin \theta = -\frac{3}{5}, \cos \theta = -\frac{4}{5} \]

- **Solution:**
  
  \[ \sin 2\theta = 2\sin \theta \cos \theta = 2 \left( -\frac{3}{5} \right) \left( -\frac{4}{5} \right) = \frac{24}{25} \]
  
  \[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left( -\frac{4}{5} \right)^2 - \left( -\frac{3}{5} \right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25} \]

### Writing in Math

**How can trigonometric functions be used to describe music?**

Include the following in your answer:

- a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next, and
- an explanation of what happens to the period of the function as you move from the 1st harmonic to the (n + 1)th harmonic.

### Enrichment, p. 872

**Alternating Current**

The figure at the right represents an alternating current generator. A rectangular coil of wire is suspended between the poles of a magnet. As the coil is rotated at a steady speed, it generates an alternating current.

At point A, the coil is parallel to the magnetic field. Therefore, no current is generated. However, at point B, the motion of A is perpendicular to the magnetic field. This induces an electromotive force in the coil. According to Faraday’s law, the coil will have an alternating current. Then, the graph of the current of an alternating current generator is closely related to the sine curve.

The maximum current may be given by the formula:

\[ i = \frac{E}{2\pi f} \]

where \( E \) is the maximum electromotive force, \( f \) is the frequency, and \( i \) is the maximum current.
Mixed Review

Find the exact value of each expression. (Lesson 14-5)

50. \( \cos 15^\circ \frac{\sqrt{6} + \sqrt{2}}{4} \)
51. \( \sin 15^\circ \frac{\sqrt{6} - \sqrt{2}}{4} \)
52. \( \sin (-135^\circ) \frac{1}{2} \)
53. \( \cos 150^\circ \frac{1}{2} \)
54. \( \sin 105^\circ \frac{\sqrt{6} + \sqrt{2}}{4} \)
55. \( \cos (-300^\circ) \frac{1}{2} \)

Verify that each of the following is an identity. (Lesson 14-4)

See pp. 811A–811N.

56. \( \cot^2 \theta - \sin^2 \theta = \frac{\cos^2 \theta \sec^2 \theta - \sin^2 \theta}{\sin^2 \theta \csc^2 \theta} \)
57. \( \cos \theta (\cos \theta + \cot \theta) = \cot \theta \cos (\theta + 1) \) See pp. 811A–811N.

EARTHQUAKE For Exercises 58 and 59, use the following information.
The magnitude of an earthquake \( M \) measured on the Richter scale is given by \( M = \log_{10} x \), where \( x \) represents the amplitude of the seismic waves causing ground motion. (Lesson 10-2)

58. How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?
59. The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock? \( 10^{1.5} \) or about 316 times as great.

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each equation.
(To review solving equations using the Zero Product Property, see Lesson 6-3.)

60. \( (x + 6)(x - 5) = 0 \)
   \[ \begin{align*}
   61. \quad (x - 1)(x + 1) &= 0 \\
   62. \quad x(x + 2) &= 0 \\
   63. \quad (2x - 5)(x + 2) &= 0 \\
   64. \quad (2x + 1)(2x - 1) &= 0
   \end{align*} \]

65. \( x^2(2x + 1) = 0 \)

66. \( (x + 6)(x - 5) = 0 \)

67. \( (x - 1)(x + 1) = 0 \)

68. \( x(x + 2) = 0 \)

69. \( (2x - 5)(x + 2) = 0 \)

70. \( (2x + 1)(2x - 1) = 0 \)

71. \( x^2(2x + 1) = 0 \)

72. \( \sin 15^\circ \frac{\sqrt{6} + \sqrt{2}}{4} \)
73. \( \sin 15^\circ \frac{\sqrt{6} - \sqrt{2}}{4} \)
74. \( \sin (-135^\circ) \frac{1}{2} \)
75. \( \cos 150^\circ \frac{1}{2} \)
76. \( \sin 105^\circ \frac{\sqrt{6} + \sqrt{2}}{4} \)
77. \( \cos (-300^\circ) \frac{1}{2} \)

Verify that each of the following is an identity. (Lessons 14-4 and 14-5) 4-6. See pp. 811A–811N.

4. \( \sin (90^\circ + \theta) = \cos \theta \)
5. \( \cos \left( \frac{3\pi}{2} - \theta \right) = -\sin \theta \)
6. \( \sin (\theta + 30^\circ) + \cos (\theta + 60^\circ) = \cos \theta \)

Find the exact value of each expression by using the double-angle or half-angle formulas. (Lesson 14-6)

7. \( \sin 2\theta \) if \( \cos \theta = -\frac{\sqrt{3}}{2} \)
   \[ 180^\circ < \theta < 270^\circ \]
8. \( \cos \frac{\theta}{2} \) if \( \sin \theta = -\frac{9}{41} \)
   \[ 270^\circ \leq \theta < 360^\circ \]
9. \( \sin 165^\circ \frac{\sqrt{2} - \sqrt{3}}{2} \)
10. \( \cos \frac{5\pi}{8} \frac{\sqrt{2} - \sqrt{3}}{2} \)

Answer (p. 796)

47. Sample answer: The sound waves associated with music can be modeled using trigonometric functions. Answers should include the following information.
   - In moving from one harmonic to the next, the number of vibrations that appear as sine waves increases by 1.
   - The period of the function as you move from the \( n \)th harmonic to the \( (n + 1) \)th harmonic decreases from \( \frac{2\pi}{n} \) to \( \frac{2\pi}{n + 1} \).

Online Lesson Plans

USA TODAY Education’s Online site offers resources and interactive features connected to each day’s newspaper. Experience TODAY, USA TODAY’s daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.

4 Assess

Open-Ended Assessment

Writing Have students write their own problems like Examples 2 and 3, and have them write an explanation of how to use a double-angle or half-angle formula to solve their examples.

Getting Ready for Lesson 14-7

PREREQUISITE SKILL In Lesson 14-7, students will solve trigonometric equations using the Zero Product Property. Use Exercises 60–65 to determine your students’ familiarity with the Zero Product Property.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 14-4 through 14-6. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 14-5 and 14-6) is available on p. 894 of the Chapter 14 Resource Masters.
Approximate Solutions In Example 1, approximate solutions can also be found by using the Trace feature. In most situations however, the Intersect feature will give more accurate solutions.

Teach

- You might wish to demonstrate the technique shown in Example 1 by first using a simple quadratic equation like \( x^2 = 9 \), whose solutions students will readily know. Graph \( y = x^2 \) and \( y = 9 \) to see that they intersect at two points, where \( x = -3 \) and \( x = 3 \), just as students will expect.
- Remind students that the solutions of the equation are the \( x \) values of the points of intersection, not the \( y \) values.
- If the expression on the right side of an equation is just 0 (as in Exercises 3 and 6), you can graph the function for the left side of the equation and then just use the Zero feature on the CALC menu to find approximate solutions.

Assess

In Exercises 3–4, check to see that students can explain why there are no real solutions. In Exercise 6, make sure students found all four possible values. Students who found fewer than four are not using the correct domain for \( x \).
**What You’ll Learn**

- Solve trigonometric equations.
- Use trigonometric equations to solve real-world problems.

**Vocabulary**

- trigonometric equation

**How can trigonometric equations be used to predict temperature?**

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function

\[ T = 11.56 \sin (0.4516x - 1.641) + 80.89 \]

where \( T \) represents the average daily high temperature in degrees Fahrenheit and \( x \) represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

**SOLVE TRIGONOMETRIC EQUATIONS** You have seen that trigonometric identities are true for all values of the variable for which the equation is defined. However, most trigonometric equations, like some algebraic equations, are true for some but not all values of the variable.

**Example 1** Solve Equations for a Given Interval

Find all solutions of each equation for the given interval.

a. \( \cos^2 \theta = 1; 0^\circ \leq \theta < 360^\circ \)

\[
\cos^2 \theta = 1 \quad \text{Original equation}
\]

\[
\cos^2 \theta - 1 = 0 \quad \text{Solve for 0.}
\]

\[
(\cos \theta + 1)(\cos \theta - 1) = 0 \quad \text{Factor.}
\]

Now use the Zero Product Property.

\[
\cos \theta + 1 = 0 \quad \text{or} \quad \cos \theta - 1 = 0
\]

\[
\cos \theta = -1 \quad \cos \theta = 1
\]

\[
\theta = 180^\circ \quad \theta = 0^\circ
\]

The solutions are \( 0^\circ \) and \( 180^\circ \).

b. \( \sin 2\theta = 2 \cos \theta; 0 \leq \theta < 2\pi \)

\[
\sin 2\theta = 2 \cos \theta \quad \text{Original equation}
\]

\[
2 \sin \theta \cos \theta = 2 \cos \theta \quad \sin 2\theta = 2 \sin \theta \cos \theta
\]

\[
2 \sin \theta \cos \theta - 2 \cos \theta = 0 \quad \text{Solve for 0.}
\]

\[
2 \cos \theta (\sin \theta - 1) = 0 \quad \text{Factor.}
\]

(continued on the next page)
Building on Prior Knowledge

In Lesson 6-3, students learned to use the Zero Product Property to solve equations. In this lesson, students will use the Zero Product Property to solve trigonometric equations.

SOLVE TRIGONOMETRIC EQUATIONS

In-Class Examples

1. Find all solutions of each equation for the given interval.
   a. \(2 \cos^2 \theta - 1 = \sin \theta\); \(0^\circ < \theta \leq 360^\circ\), \(30^\circ\), \(150^\circ\), \(270^\circ\)
   b. \(\sin \theta = \sin 2\theta; 0 < \theta \leq 2\pi\)

   \(\frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{3}, 2\pi\)

   Teaching Tip In Example 2, show students how to look for patterns in the solution of part a. Students should look for pairs of solutions that differ by exactly \(\pi\) or \(2\pi\).

2. a. Solve \(2 \sin \theta \cos \theta = \cos \theta\) for all values of \(\theta\) if \(\theta\) is measured in radians.
   \(\frac{\pi}{6} + 2k\pi, \frac{\pi}{2} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{3\pi}{2} + 2k\pi\), where \(k\) is any integer

   b. Solve \(\cos \theta = -\cos 2\theta\) for all values of \(\theta\) if \(\theta\) is measured in degrees. \(60^\circ + k \cdot 180^\circ\), \(180^\circ + k \cdot 360^\circ\), \(300^\circ + k \cdot 360^\circ\), where \(k\) is any integer

   Study Tip

   Expressing Solutions as Multiples
   The expression \(90^\circ + k \cdot 180^\circ\) includes \(270^\circ\) and its multiples, so it is not necessary to list them separately.

Example 2

Solve Trigonometric Equations

a. Solve \(2 \sin \theta = -1\) for all values of \(\theta\) if \(\theta\) is measured in radians.

   \(2 \sin \theta = -1\) Original equation
   \(\sin \theta = -\frac{1}{2}\) Divide each side by 2.

   Look at the graph of \(y = \sin \theta\) to find solutions of \(\sin \theta = -\frac{1}{2}\).

   The solutions are \(\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}\), and so on, and \(-\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, -\frac{23\pi}{6}\), and so on. The only solutions in the interval \(0\) to \(2\pi\) are \(\frac{7\pi}{6}\) and \(\frac{11\pi}{6}\). The period of the sine function is \(2\pi\) radians. So the solutions can be written as \(\frac{7\pi}{6} + 2k\pi\) and \(\frac{11\pi}{6} + 2k\pi\), where \(k\) is any integer.

b. Solve \(\cos 2\theta + \cos \theta + 1 = 0\) for all values of \(\theta\) if \(\theta\) is measured in degrees.

   \(\cos 2\theta + \cos \theta + 1 = 0\) Original equation
   \(2 \cos^2 \theta - 1 + \cos \theta + 1 = 0\)
   \(2 \cos^2 \theta + \cos \theta = 0\) Simplify.
   \(\cos \theta (2 \cos \theta + 1) = 0\) Factor.

   Solve for \(\theta\) in the interval \(0^\circ\) to \(360^\circ\).

   \(\cos \theta = 0\) or \(2 \cos \theta + 1 = 0\)

   \(\theta = 90^\circ\) or \(270^\circ\) or \(\theta = -\frac{1}{2}\)

   \(\theta = 120^\circ\) or \(240^\circ\)

   The solutions are \(90^\circ + k \cdot 180^\circ\), \(120^\circ + k \cdot 360^\circ\), and \(240^\circ + k \cdot 360^\circ\).

   If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.
Example 3 Solve Trigonometric Equations Using Identities

Solve $\cos \theta \tan \theta - \sin^2 \theta = 0$.

Original equation

$\cos \theta \tan \theta - \sin^2 \theta = 0$

$\cos \theta \left( \frac{\sin \theta}{\cos \theta} \right) - \sin^2 \theta = 0$

$\tan \theta - \sin^2 \theta = 0$ Multiply.

$\sin \theta \cos \theta (1 - \sin \theta) = 0$ Factor.

$\sin \theta = 0$ or $1 - \sin \theta = 0$

$\theta = 0^\circ$, $180^\circ$, or $360^\circ$ $\sin \theta = 1$

$\theta = 90^\circ$

CHECK

$\cos \theta \tan \theta - \sin^2 \theta = 0$

$\cos 0^\circ \tan 0^\circ - \sin^2 0^\circ = 0 \theta = 0^\circ$ $\cos 180^\circ \tan 180^\circ - \sin^2 180^\circ \neq 0 \theta = 180^\circ$

$1 \cdot 0 - 0 \neq 0$ $-1 \cdot 0 - 0 \neq 0$

$0 = 0 \checkmark$ $0 = 0 \checkmark$

$\cos 360^\circ \tan 360^\circ - \sin^2 360^\circ \neq 0 \theta = 360^\circ$ $\cos 90^\circ \tan 90^\circ - \sin^2 90^\circ \neq 0 \theta = 90^\circ$

$1 \cdot 0 - 0 \neq 0$ $0 = 0 \checkmark$

$0 = 0 \checkmark$ $\tan 90^\circ$ is undefined.

Thus, $90^\circ$ is not a solution.

The solution is $0^\circ + k \cdot 180^\circ$.

Example 4 Determine Whether a Solution Exists

Solve $3 \cos 2\theta - 5 \cos \theta = 1$.

Original equation

$3 \cos 2\theta - 5 \cos \theta = 1$

$3(2 \cos^2 \theta - 1) - 5 \cos \theta = 1$ Multiply.

$6 \cos^2 \theta - 3 - 5 \cos \theta = 1$ Subtract 1 from each side.

$(3 \cos \theta - 4)(2 \cos \theta + 1) = 0$ Factor.

$3 \cos \theta - 4 = 0$ or $2 \cos \theta + 1 = 0$

$3 \cos \theta = 4$ $2 \cos \theta = -1$

$\cos \theta = \frac{4}{3}$ $\cos \theta = -\frac{1}{2}$

Not possible since $\theta = 120^\circ$ or $240^\circ$ $\cos \theta$ cannot be greater than 1.

Thus, the solutions are $120^\circ + k \cdot 360^\circ$ and $240^\circ + k \cdot 360^\circ$.

www.algebra2.com/extra_examples Lesson 14-7 Solving Trigonometric Equations 801

DAILY Intervention

Interpersonal As students work through this lesson, have them create a class list on the chalkboard that identifies common errors they made. Encourage students to add suggestions for how to avoid their errors. For example, one common error is having one’s calculator set to degrees when it needs to be set to radians for a problem, and vice versa.
USE TRIGONOMETRIC EQUATIONS

In-Class Example

\[
\text{Example 5 Use a Trigonometric Equation}
\]

GARDENING Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight \( H \) in her town can be represented by \( H = 11.45 + 6.5 \sin (0.0168d - 1.333) \), where \( d \) is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?

\[
\begin{align*}
H & = 11.45 + 6.5 \sin (0.0168d - 1.333) \quad \text{Original equation} \\
14 & = 11.45 + 6.5 \sin (0.0168d - 1.333) \quad H = 14 \\
2.55 & = 6.5 \sin (0.0168d - 1.333) \quad \text{Subtract 11.45 from each side.} \\
0.392 & = \sin (0.0168d - 1.333) \quad \text{Divide each side by 6.5.} \\
0.403 & = 0.0168d - 1.333 \\
1.736 & = 0.0168d \quad \text{Add 1.333 to each side.} \\
103.333 & = d \quad \text{Divide each side by 0.0168.}
\end{align*}
\]

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

Check for Understanding

1. Tell why the equation \( \sec \theta = 0 \) has no solutions.
2. Explain why the number of solutions to the equation \( \sin \theta = \frac{\sqrt{3}}{2} \) is infinite.
3. OPEN ENDED Write an example of a trigonometric equation that has no solution.

Find all solutions of each equation for the given interval.

\[
\begin{align*}
4. & \; 4 \cos^2 \theta = 1; \; 0^\circ \leq \theta < 360^\circ \\
& \quad 60^\circ, 120^\circ, 240^\circ, 300^\circ \\
& \quad 135^\circ, 225^\circ \\
& \quad 6. & \; \sin 2\theta = \cos \theta; \; 0 \leq \theta < 2\pi \\
& \quad \frac{\sin \theta}{\cos \theta} = 1 + \cos \theta \\
& \quad 3 \sin^2 \theta - \cos^2 \theta = 0; \; 0 \leq \theta < \frac{\pi}{2} \\
& \quad 7. & \; \sin 2\theta + \cos \theta = 0 \quad \sin \theta + \sin \theta \cos \theta = 0 \\
& \quad 8. & \; \cos 2\theta = \cos \theta \\
& \quad \sin \theta = 1 + \cos \theta \\
& \quad 9. & \; 2 \cos^2 \theta + 2 = 5 \cos \theta \\
& \quad \sin \theta = 1 + \cos \theta \\
& \quad 10. & \; 90^\circ + k \cdot 360^\circ, 180^\circ + k \cdot 360^\circ \\
& \quad 60^\circ + k \cdot 360^\circ, 300^\circ + k \cdot 360^\circ \\
& \quad 11. & \; \sin^2 \theta - 3 \sin \theta - 2 = 0 \\
& \quad 9 \cos^2 \theta + 3 \sin \theta - 3 = 0 \\
& \quad 12. & \; \sin \theta = 1 + \cos \theta
\end{align*}
\]

14. PHYSICS According to Snell’s law, the angle at which light enters water \( \alpha \) is related to the angle at which light travels in water \( \beta \) by the equation \( \sin \alpha = 1.33 \sin \beta \). At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water?

\[
\begin{align*}
\alpha & = 31.3^\circ
\end{align*}
\]

Answers

1. Sample answer: If \( \sec \theta = 0 \) then \( \frac{1}{\cos \theta} = 0 \). Since no value of \( \theta \) makes \( \frac{1}{\cos \theta} = 0 \), there are no solutions.

2. Sample answer: The function is periodic with two solutions in each of its infinite number of periods.

3. Sample answer: \( \sin \theta = 2 \)

4. \( \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi \)

5. \( 24. \; \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi \)

6. \( 5\pi \)

7. \( 27. \; \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi \)

8. \( 28. \; \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \)

9. \( 29. \; \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \)
15. \(2 \cos \theta - 1 = 0; 0^\circ \leq \theta < 360^\circ\)
16. \(2 \sin \theta = -\sqrt{3}; 180^\circ < \theta < 360^\circ\)
17. \(4 \sin^2 \theta = 1; 180^\circ < \theta < 360^\circ\)
18. \(4 \cos^2 \theta = 3; 0^\circ \leq \theta < 360^\circ\)
19. \(2 \cos^2 \theta + \sin \theta + 1 = 0; \pi < \theta < 2\pi\)
20. \(2 \sin^2 \theta - 1 = \cos^2 \theta; 0 \leq \theta < \pi\)
21. \(2 \sin^2 \theta + \sin \theta = 0; \pi < \theta < 2\pi\)
22. \(2 \cos^2 \theta = -\cos \theta; 0 \leq \theta < 2\pi\)

Solve each equation for all values of \(\theta\) if \(\theta\) is measured in radians.
23. \(\cos 2\theta + 3 \cos \theta - 1 = 0\)
24. \(2 \sin^2 \theta - \cos \theta - 1 = 0\)
25. \(\cos^2 \theta - \frac{5}{2} \cos \theta - \frac{3}{2} = 0\)
26. \(\cos \theta = 3 \cos \theta - 2\)
27. \(4 \cos^2 \theta - 4 \cos \theta + 1 = 0\)
28. \(2 \cos 2\theta = -1 - \sin \theta\)
29-34. See margin.

Solve each equation for all values of \(\theta\) if \(\theta\) is measured in degrees.
29. \(\sin \theta = \cos \theta\)
30. \(\tan \theta = \sin \theta\)
31. \(\sin^2 \theta - 2 \sin \theta - 3 = 0\)
32. \(4 \sin^2 \theta - 4 \sin \theta + 1 = 0\)
33. \(\tan^2 \theta - 3 \tan \theta = 0\)
34. \(3 \cos^2 \theta - 2 \cos \theta - 2 = 0\)

35. \(\sin^2 \theta + \cos 2\theta - \cos \theta = 0\)
36. \(2 \sin^2 \theta - 3 \sin \theta - 2 = 0\)
37. \(\sin^2 \theta = \cos^2 \theta - 1\)
38. \(2 \cos^2 \theta + \cos \theta = 0\)
39. \(\frac{3}{2} + \cos \theta = 1\)
40. \(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}\)

41. \(S = 352\) or \(S = 352 \cot \theta\)

For Exercises 41 and 42, use the information shown.
41. The length of the shadow \(S\) of the International Peace Memorial at Put-In-Bay, Ohio, depends upon the angle of inclination of the Sun, \(\theta\). Express \(S\) as a function of \(\theta\).
42. Find the angle of inclination \(\theta\) that will produce a shadow 560 feet long. About 32°

For Exercises 43 and 44, use the following information.
For a short time after a wave is created by a boat, the height of the wave can be modeled using \(y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}\), where \(h\) is the maximum height of the wave in feet, \(P\) is the period in seconds, and \(t\) is the propagation of the wave in seconds.
43. If \(h = 3\) and \(P = 2\) seconds, write the equation for the wave and draw its graph over a 10-second interval. See pp. 811A–811N.
44. How many times over the first 10 seconds does the graph predict the wave to be one foot high? 10

Answers
29. 45° + k • 180°
30. 0° + k • 180°
31. 270° + k • 360°
32. 30° + k • 360°, 150° + k • 360°
33. 0° + k • 180°, 60° + k • 180°
34. 120° + k • 360°, 240° + k • 360°

Enrichment, p. 878

Families of Curves
Use these graphs for the problems below.

The Family \(f = a\)

The Family \(f = -a\)
Open-Ended Assessment

Speaking Have students show you a trigonometric equation they solved and explain step by step how they performed each step of their computation.

Assessment Options

Quiz (Lesson 14-7) is available on p. 894 of the Chapter 14 Resource Masters.

Answer

46. Sample answer: Temperatures are cyclic and can be modeled by trigonometric functions. Answers should include the following information.

- A temperature could occur twice in a given period such as when the temperature rises in the spring and falls in autumn.

45. **CRITICAL THINKING** Computer games often use transformations to distort images on the screen. In one such transformation, an image is rotated counterclockwise using the equations \(x' = x \cos \theta - y \sin \theta\) and \(y' = x \sin \theta + y \cos \theta\). If the coordinates of an image point are \((3, 4)\) after a 60° rotation, what are the coordinates of the preimage point? \((4.964, -0.598)\)

46. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See margin.

How can trigonometric equations be used to predict temperature?

Include the following in your answer:

- an explanation of why the sine function can be used to model the average daily temperature,
- an explanation of why, during one period, you might find a specific average temperature twice.

47. Which of the following is not a possible solution of \(0 = \sin \theta + \cos \theta \tan^2 \theta\)? 

- **A** \(\frac{3\pi}{4}\)
- **B** \(\frac{7\pi}{4}\)
- **C** \(2\pi\)
- **D** \(\frac{5\pi}{2}\)

48. The graph of the equation \(y = 2 \cos \theta\) is shown. Which is a solution for \(2 \cos \theta = 1\)?

- **A** \(\frac{8\pi}{3}\)
- **B** \(\frac{13\pi}{3}\)
- **C** \(\frac{10\pi}{3}\)
- **D** \(\frac{15\pi}{3}\)

---

**Maintain Your Skills**

**Mixed Review**

Find the exact value of \(\sin 2\theta\), \(\cos 2\theta\), \(\sin \frac{\theta}{2}\) and \(\cos \frac{\theta}{2}\) for each of the following.

49. \(\sin \theta = \frac{3}{5} \); \(0^\circ < \theta < 90^\circ\)
50. \(\cos \theta = \frac{1}{2} \); \(0^\circ < \theta < 90^\circ\)
51. \(\cos \theta = \frac{6}{5} \); \(0^\circ < \theta < 90^\circ\)
52. \(\sin \theta = \frac{4}{5} \); \(0^\circ < \theta < 90^\circ\)

Find the exact value of each expression.

53. \(\sin 240^\circ \)
54. \(\cos 315^\circ \)

55. Solve \(\triangle ABC\). Round measures of sides and angles to the nearest tenth.

- \(b = 11.0\), \(c = 12.2\), \(m\angle C = 78^\circ\)

---

**WebQuest Internet Project**

*Trig Class Angles for Lessons in Lit*

It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

[www.algebra2.com/webquest](http://www.algebra2.com/webquest)
Vocabulary and Concept Check

amplitude (p. 763)  midline (p. 771)  trigonometric identity (p. 777)
double-angle formula (p. 791)  phase shift (p. 769)  vertical shift (p. 771)
half-angle formula (p. 793)  trigonometric equation (p. 799)

Choose the correct letter that best matches each phrase.

1. horizontal translation of a trigonometric function  h
2. a reference line about which a graph oscillates  b
3. vertical translation of a trigonometric function  d
4. the formula used to find \( \cos \frac{2\theta}{2} \)  f
5. \( \sin 2\theta = 2 \sin \theta \cos \theta \)  e
6. a measure of how long it takes for a graph to repeat itself  c
7. \( \cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \)  g
8. the absolute value of half the difference between the maximum and minimum values of a periodic function  a

Lesson-by-Lesson Review

14-1

See pages 762-764.

**Graphing Trigonometric Functions**

**Concept Summary**

- For trigonometric functions of the form \( y = a \sin b\theta \) and \( y = a \cos b\theta \), the amplitude is \( |a| \), and the period is \( \frac{360^\circ}{|b|} \) or \( \frac{2\pi}{|b|} \).
- The period of \( y = a \tan b\theta \) is \( \frac{180^\circ}{|b|} \) or \( \frac{\pi}{|b|} \).

**Example**

Find the amplitude and period of \( y = 2 \cos 4\theta \).

Then graph the function.

The amplitude is \( 2 \) or 2.

The period is \( \frac{360^\circ}{4} \) or 90°.

Use the amplitude and period to graph the function.

**Exercises**

Find the amplitude, if it exists, and period of each function. Then graph each function. See Example 1 on page 765.

9. \( y = -\frac{1}{2} \cos \theta \)  
10. \( y = 4 \sin 2\theta \)  
11. \( y = \sin \frac{1}{2} \theta \)  
12. \( y = 5 \sec \theta \)  
13. \( y = \frac{1}{2} \csc \frac{\theta}{3} \)  
14. \( y = \tan 4\theta \)

www.algebra2.com/vocabulary_review

Chapter 14  Study Guide and Review  805

**Foldables**

Have students look through the chapter to make sure they have included notes and examples of graphs for each lesson in this chapter in their Foldable.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.

**Vocabulary and Concept Check**

- This alphabetical list of vocabulary terms in Chapter 14 includes a page reference where each term was introduced.
- **Assessment** A vocabulary test/review for Chapter 14 is available on p. 892 of the Chapter 14 Resource Masters.

**Lesson-by-Lesson Review**

For each lesson,

- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

**Vocabulary PuzzleMaker**

ELL The Vocabulary PuzzleMaker software improves students’ mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

**MindJogger Videoquizzes**

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)
14–2 Translations of Trigonometric Graphs

Concept Summary

- For trigonometric functions of the form \( y = a \sin (b \theta - h) \), \( y = a \cos (b \theta - h) \), and \( y = a \tan (b \theta - h) \), the phase shift is to the right when \( h > 0 \) and to the left when \( h < 0 \).

- For trigonometric functions of the form \( y = a \sin (b \theta - h) + k \), \( y = a \cos (b \theta - h) + k \), and \( y = a \tan (b \theta - h) + k \), the vertical shift is up when \( k > 0 \) and down when \( k < 0 \).

Example

State the vertical shift, amplitude, period, and phase shift of \( y = 3 \sin \left(2 \left(\frac{\theta - \pi}{2}\right)\right) - 2 \).

Then graph the function.

Identify the values of \( k, a, b, \) and \( h \).

- \( k = -2 \), so the vertical shift is \(-2\).
- \( a = 3 \), so the amplitude is 3.
- \( b = 2 \), so the period is \( \frac{2\pi}{2} \) or \( \pi \).
- \( h = \frac{\pi}{2} \), so the phase shift is \( \frac{\pi}{2} \) to the right.

Exercises

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function. See Example 3 on page 772.

15. \( y = \frac{1}{2} \sin \left(2 \left(\theta - 60^\circ\right)\right) - 1 \)
16. \( y = 2 \tan \left(\frac{1}{4} \left(\theta - 90^\circ\right)\right) + 3 \)
17. \( y = 3 \sec \left(\frac{1}{2} \left(\theta + \frac{\pi}{4}\right)\right) + 1 \)
18. \( y = \frac{1}{3} \cos \left(\frac{1}{3} \left(\theta - \frac{2\pi}{3}\right)\right) - 2 \)

14–3 Trigonometric Identities

Concept Summary

- Quotient Identities: \( \tan \theta = \frac{\sin \theta}{\cos \theta} \), \( \cot \theta = \frac{\cos \theta}{\sin \theta} \)
- Reciprocal Identities: \( \csc \theta = \frac{1}{\sin \theta} \), \( \sec \theta = \frac{1}{\cos \theta} \), \( \cot \theta = \frac{1}{\tan \theta} \)
- Pythagorean Identities: \( \cos^2 \theta + \sin^2 \theta = 1 \), \( \tan^2 \theta + 1 = \sec^2 \theta \), \( \cot^2 \theta + 1 = \csc^2 \theta \)

Example

Simplify \( \sin \theta \cot \theta \cos \theta \).

\[
\sin \theta \cot \theta \cos \theta = \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} = 1
\]

Multiply.

Exercises

Find the value of each expression. See Example 1 on page 778.

19. \( \cot \theta \), if \( \csc \theta = -\frac{5}{3} \); \( 270^\circ < \theta < 360^\circ \)
20. \( \sec \theta \), if \( \sin \theta = \frac{1}{2} \); \( 0^\circ \leq \theta < 90^\circ \)

Simplify each expression. See Example 2 on page 778.

21. \( \sin^2 \theta \)
22. \( \cos^2 \theta \)
23. \( \csc \theta \)
24. \( \cot \theta \)
25. \( \sec \theta \)
26. \( \cos \theta \)
27. \( \sin \theta \)
28. \( \csc \theta \)
29. \( \cos \theta \)
30. \( \sin \theta \)
31. \( \csc \theta \)
32. \( \cot \theta \)
33. \( \sec \theta \)
34. \( \cos \theta \)
35. \( \sin \theta \)
36. \( \csc \theta \)
37. \( \cot \theta \)
38. \( \sec \theta \)
39. \( \cos \theta \)
40. \( \sin \theta \)
41. \( \csc \theta \)
42. \( \cot \theta \)
43. \( \sec \theta \)
44. \( \cos \theta \)
45. \( \sin \theta \)
46. \( \csc \theta \)
47. \( \cot \theta \)
48. \( \sec \theta \)
49. \( \cos \theta \)
50. \( \sin \theta \)
51. \( \csc \theta \)
52. \( \cot \theta \)
53. \( \sec \theta \)
54. \( \cos \theta \)
55. \( \sin \theta \)
56. \( \csc \theta \)
57. \( \cot \theta \)
58. \( \sec \theta \)
59. \( \cos \theta \)
60. \( \sin \theta \)
61. \( \csc \theta \)
62. \( \cot \theta \)
63. \( \sec \theta \)
64. \( \cos \theta \)
65. \( \sin \theta \)
66. \( \csc \theta \)
67. \( \cot \theta \)
68. \( \sec \theta \)
69. \( \cos \theta \)
70. \( \sin \theta \)
### 14-4 Verifying Trigonometric Identities

**Concept Summary**
- Use the basic trigonometric identities to transform one or both sides of a trigonometric equation into the same form.

**Example**
Verify that \( \tan \theta + \cot \theta = \sec \theta \csc \theta \).

\[
\begin{align*}
\tan \theta + \cot \theta & = \sec \theta \csc \theta \\
\text{Original equation} \\
\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} & = \sec \theta \csc \theta \\
\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} & = \sec \theta \csc \theta \\
\frac{1}{\cos \theta \sin \theta} & = \sec \theta \csc \theta \\
\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} & = \sec \theta \csc \theta \\
\sec \theta \csc \theta & = \sec \theta \csc \theta \\
\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} & = \sec \theta \csc \theta \\
\sec \theta \csc \theta & = \sec \theta \csc \theta
\end{align*}
\]

**Exercises**
Verify that each of the following is an identity. See Examples 1–3 on pages 782–783. 24–27. See margin.

24. \( \sin \theta + \cos \theta = \cot \theta \csc \theta \)
25. \( \sin \theta - \cot \theta = \sec \theta \csc \theta \)
26. \( \cot \theta \sec \theta = 1 + \csc^2 \theta \)

### 14-5 Sum and Difference of Angles Formulas

**Concept Summary**
- For all values of \( \alpha \) and \( \beta \):
  \[
  \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta 
  \]
  \( \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \)

**Example**
Find the exact value of \( \sin 195^\circ \).

\[
\begin{align*}
\sin 195^\circ & = \sin (150^\circ + 45^\circ) \\
& = \sin 150^\circ \cos 45^\circ + \cos 150^\circ \sin 45^\circ \\
& = \left(\frac{1}{2}\right) \left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{2}}{2}\right) \\
& = \frac{\sqrt{2} - \sqrt{6}}{4} \\
& \text{Evaluate each expression.} \\
& \text{Simplify.}
\end{align*}
\]

**Exercises**
Find the exact value of each expression. See Example 1 on page 787.

28. \( \cos 15^\circ \frac{\sqrt{6} + \sqrt{2}}{2} \)
29. \( \cos 285^\circ \frac{\sqrt{6} - \sqrt{2}}{2} \)
30. \( \cos -150^\circ \frac{1}{2} \)
31. \( \sin 195^\circ \frac{4}{2} \)
32. \( \cos (-210^\circ) \frac{4}{\sqrt{3}} \)
33. \( \sin (-150^\circ) \frac{-\sqrt{3} - \sqrt{2}}{4} \)

Verify that each of the following is an identity. See Example 3 on page 788.

34. \( \cos (90^\circ + \theta) = -\sin \theta \)
35. \( \sin (30^\circ - \theta) = \cos (60^\circ + \theta) \)
36. \( \sin (\theta + \pi) = -\sin \theta \)
37. \( -\cos \theta = \cos (\pi + \theta) \)

---

**Answer**

24. \[
\begin{align*}
\sin \theta + \cos \theta & = \cot \theta \csc \theta \\
\sin \theta - \cos \theta & = \cot \theta \csc \theta \\
\sin \theta \cos \theta & = \cot \theta \csc \theta \\
\cos \theta + \sin \theta & = \cos \theta + \sin \theta
\end{align*}
\]
14-6 Double-Angle and Half-Angle Formulas

Concept Summary
- Double-angle formulas: \(\sin 2\theta = 2\sin \theta \cos \theta, \cos 2\theta = \cos^2 \theta - \sin^2 \theta\), \(\cos 2\theta = 1 - 2\sin^2 \theta, \cos 2\theta = 2\cos^2 \theta - 1\)
- Half-angle formulas: \(\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}\), \(\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}\)

Example
Verify that \(\csc 2\theta = \frac{\sec \theta}{2\sin \theta}\) is an identity.

\[
\begin{align*}
\csc 2\theta &= \frac{\sec \theta}{2\sin \theta} \\
\frac{1}{\sin 2\theta} &= \frac{\sec \theta}{2\sin \theta} \\
\frac{1}{\sin 2\theta} &= \frac{\cos \theta}{2\sin \theta} \\
\frac{1}{\sin 2\theta} &= \frac{1}{2\sin \theta \cos \theta} \\
2\sin \theta \cos \theta &= \sin 2\theta
\end{align*}
\]

Exercises
Find the exact values of \(\sin 2\theta, \cos 2\theta, \sin \frac{\theta}{2}\), and \(\cos \frac{\theta}{2}\) for each of the following. See Examples 1 and 2 on pages 792 and 793.

38. \(\sin \theta = \frac{1}{4}, 0^\circ < \theta < 90^\circ\)
39. \(\sin \theta = \frac{5}{12}, 180^\circ < \theta < 270^\circ\)
40. \(\cos \theta = -\frac{5}{12}, 90^\circ < \theta < 180^\circ\)
41. \(\cos \theta = \frac{12}{13}, 270^\circ < \theta < 360^\circ\)

14-7 Solving Trigonometric Equations

Concept Summary
- Solve trigonometric equations by factoring or by using trigonometric identities.

Example
Solve \(\sin 2\theta + \sin \theta = 0\) if \(0^\circ \leq \theta < 360^\circ\).

\[
\begin{align*}
\sin 2\theta + \sin \theta &= 0 \\
2\sin \theta \cos \theta + \sin \theta &= 0 \\
\sin \theta (2\cos \theta + 1) &= 0 \\
\sin \theta &= 0 \quad \text{or} \quad 2\cos \theta + 1 = 0 \\
\theta &= 0^\circ \text{ or } 180^\circ \quad \text{or} \quad 2\cos \theta = -1 \\
\theta &= 120^\circ \text{ or } 240^\circ
\end{align*}
\]

Exercises
Find all solutions of each equation for the interval \(0^\circ \leq \theta < 360^\circ\).

42. \(2\sin 2\theta = 1\) \(15^\circ, 75^\circ, 195^\circ, 255^\circ\)
43. \(2\cos^2 \theta + \sin^2 \theta = 2\cos \theta \quad 0^\circ\)

Solve each equation for all values of \(\theta\) if \(\theta\) is measured in radians.

44. \(6\sin^2 \theta - 5\sin \theta - 4 = 0\)
45. \(2\cos^2 \theta = 3\sin \theta\)
19. STANDARDIZED TEST PRACTICE

travels is found by the formula

A golf ball is hit with an initial velocity of 100 feet per second. The distance the ball

9. The (period, phase shift) of $y = 3 \sin (\theta - 60^\circ)$ is $2. A$ midline is used with a (phase shift, vertical shift) of a trigonometric function.

3. The amplitude of $y = \frac{1}{3} \cos (3\theta + 4) - 1$ is $\left(\frac{1}{3}, 3\right)$.

4. The (cosine, cosecant) has no amplitude.

Vocabulary and Concepts

Choose the correct term to complete each sentence.
1. The (period, phase shift) of $y = 3 \sin (\theta - 60^\circ)$ is $2. A$ midline is used with a (phase shift, vertical shift) of a trigonometric function.

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function. 5–6. See margin.

5. $y = \frac{2}{3} \sin 2\theta + 5$
6. $y = 4 \cos \left[\frac{1}{2}(\theta + 30^\circ)\right] - 1$

Find the value of each expression.

7. $\tan \theta$, if $\sin \theta = \frac{1}{2}$$; 90^\circ < \theta < 180^\circ \quad \frac{-\sqrt{3}}{3}$
8. $\sec \theta$, if $\cot \theta = \frac{3}{4}$$; 180^\circ < \theta < 270^\circ \quad -\frac{5}{3}$

Verify that each of the following is an identity. 9–12. See pp. 811A–811N.

9. $(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta$
10. $\frac{\cos \theta}{1 - \sin^2 \theta} = \sec \theta$
11. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$
12. $\frac{1 + \tan^2 \theta}{\cos^2 \theta} = \sec^2 \theta$

Find the exact value of each expression. 13. $\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}$
14. $\sqrt{\frac{2}{\sqrt{\frac{2}{2}}} - \sqrt{\frac{2}{2}}}$
15. $\sqrt{\frac{3}{2}}$
13. $\sin 285^\circ$
14. $\sin 345^\circ$
16. $\cos 67.5^\circ$
17. $\cos 285^\circ$
18. $\cos 240^\circ$
19. $\sec 2\theta = \cos \theta$
20. $\cos 2\theta = \cos \theta$
21. $\cos 2\theta + \sin \theta = 1$
22. $\sin \theta = \tan \theta$

GOLF For Exercises 23 and 24, use the following information.

A golf ball is hit with an initial velocity of 100 feet per second. The distance the ball travels is found by the formula $d = \frac{v_0^2}{g} \sin 2\theta$, where $v_0$ is the initial velocity, $g$ is the acceleration due to gravity, 32 feet per second squared, and $\theta$ is the measurement of the angle that the path of the ball makes with the ground.

23. Find the distance that the ball travels if the angle between the path of the ball and the ground measures 60°. 270.6 ft
24. If a ball travels 312.5 feet, what was the angle the path of the ball made with the ground to the nearest degree? 45°

25. STANDARDIZED TEST PRACTICE Identify the equation of the graphed function. B

A. $y = 3 \cos 2\theta$
B. $y = \frac{1}{3} \cos 2\theta$
C. $y = 3 \cos \frac{1}{2}\theta$
D. $y = \frac{1}{3} \cos \frac{1}{2}\theta$

www.algebra2.com/chapter_test

Portfolio Suggestion

Introduction In mathematics, trigonometric functions can be used to model real-world problems.

Ask Students Find a real-world problem modeled in this chapter that interests you and show how you solved it. Explain how the function models the real-world problem and what could be gained by understanding the real-world problem better. Place your work in your portfolio.

Assessment Options

Vocabulary Test A vocabulary test/review for Chapter 14 can be found on p. 892 of the Chapter 14 Resource Masters.

Chapter Tests There are six Chapter 14 Tests and an Open-Ended Assessment task available in the Chapter 14 Resource Masters.

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Open-Ended Assessment

Performance tasks for Chapter 14 can be found on p. 891 of the Chapter 14 Resource Masters. A sample scoring rubric for these tasks appears on p. A28.

Unit 5 Test A unit test/review can be found on pp. 899–900 of the Chapter 14 Resource Masters.

End-of-Year Tests A Second Semester Test for Chapters 8–14 and a Final Test for Chapters 1–14 can be found on pp. 901–910 of the Chapter 14 Resource Masters.

TestCheck and Worksheet Builder

This networkable software has three modules for assessment.

- Worksheet Builder to make worksheets and tests.
- Student Module to take tests on-screen.
- Management System to keep student records.
These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the Chapter 14 Resource Masters.

**Standardized Test Practice Student Recording Sheet, p. A1**

**Part 1: Multiple Choice**

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. Which of the following is not equal to $3.5 \times 10^{-22}$? D
   - A: $\frac{35}{1000}$
   - B: 0.035
   - C: $\frac{7}{200}$
   - D: $(0.5)(0.007)$

2. The sum of five consecutive odd integers is 55. What is the sum of the greatest and least of these integers? B
   - A: 11
   - B: 22
   - C: 26
   - D: 30

3. If 8 bananas cost $a$ cents and 6 oranges cost $b$ cents, what is the cost of 2 bananas and 2 oranges in terms of $a$ and $b$? D
   - A: $\frac{ab}{12}$
   - B: $3a + b$
   - C: $3a + 4b$
   - D: $3a + 4b \div 12$

4. A bag contains 16 peppermint candies, 10 butterscotch candies, and 8 cherry candies. Emma chooses one piece at random, puts it in her pocket, and then repeats the process. If she has chosen 3 peppermint candies, 2 butterscotch candies, and 1 cherry candy, what is the probability that the next piece of candy she chooses will be cherry? C
   - A: $\frac{7}{34}$
   - B: $\frac{8}{34}$
   - C: $\frac{1}{4}$
   - D: $\frac{3}{4}$

5. What is the value of $\frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{3}}$? B
   - A: $-\sqrt{3}$
   - B: -1
   - C: $-\frac{\sqrt{3}}{3}$
   - D: 1

6. In right triangle QRS, what is the value of $\tan \theta$? D

7. What is the value of $\sin \left( \cos^{-1} \frac{1}{3} \right)$? B
   - A: $\frac{2}{3}$
   - B: $\frac{2\sqrt{2}}{3}$
   - C: $\frac{\sqrt{2}}{3}$
   - D: $\frac{\sqrt{6}}{3}$

8. What is the least positive value for $x$ where $y = \sin 2x$ reaches its minimum? C
   - A: $\frac{\pi}{2}$
   - B: $\pi$
   - C: $\frac{3\pi}{4}$
   - D: $\frac{3\pi}{2}$

9. Which of the following is equivalent to $\sin^2 \theta + \cos^2 \theta$? A
   - A: $\cos^2 \theta$
   - B: $\sin^2 \theta$
   - C: $\tan^2 \theta$
   - D: $\sin^2 \theta + 1$

10. If $\cos \theta = -\frac{1}{2}$ and $\theta$ is in Quadrant II, what is the value of $\sin 2\theta$? D
    - A: $\frac{1}{2}$
    - B: $-\frac{1}{2}$
    - C: $\frac{\sqrt{3}}{2}$
    - D: $-\frac{\sqrt{3}}{2}$

**Additional Practice**

See pp. 897–898 in the Chapter 14 Resource Masters for additional standardized test practice.

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**TestCheck and Worksheet Builder**

Special banks of standardized test questions similar to those on the SAT, ACT, TIMSS 8, NAEP 8, and Algebra 1 End-of-Course tests can be found on this CD-ROM.
Part 2  Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. If $k$ is a positive integer, and $7k + 3$ equals a prime number that is less than 50, then what is one possible value of $7k + 3$? 17 or 31

12. It costs $8 to make a book. The selling price will include an additional 200%. What will be the selling price? $24

13. The mean of seven numbers is 0. The sum of three of the numbers is $9. What is the sum of the remaining four numbers? 9

14. If $4a - 6b = 0$ and $c = 9b$, what is the ratio of $a$ to $c$? 1/6

15. What is the value of $x$ if $\frac{3x^3 - 3}{\sqrt{81}} = 3$? 2

16. The ages of children at a party are 6, 7, 6, 6, 7, 7, 8, 6, 7, 8, 9, 7, and 7. Let $N$ represent the median of their ages and $m$ represent the mode. What is $N - m$? 0

17. In the figure below, CEFG is a square, ABD is a right triangle, $D$ is the midpoint of side $CE$, $H$ is the midpoint of side $CG$, and $C$ is the midpoint of side $BD$. BCDE is a line segment, and $AHD$ is a line segment. If the measure of the area of square CEFG is 16, what is the measure of the area of quadrilateral $ABCH$? 6

18. A line with a slope of $\frac{3}{8}$ passes through points $(6, 4n)$ and $(0, n)$. What is the value of $n$? 3/4

19. If $\sin 60^\circ = \frac{\sqrt{3}}{2}$, what is the value of $\sin^2 30^\circ + \cos^2 30^\circ$? 1

Part 3  Quantitive Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

A the quantity in Column A is greater,
B the quantity in Column B is greater,
C the two quantities are equal, or
D the relationship cannot be determined from the information given.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

20. the length of a diagonal of a square whose area is 100 the length of a diagonal of a 6 x 8 rectangle

21. $\sqrt{c} = c^2 + \frac{2}{c} - 2$

22. $w = 2x, x = \frac{1}{2}w$

23. $(a + b)^2 = a^2 + b^2$

24. $(a - b)^2 = a^2 + b^2$
4. amplitude: $\frac{1}{2}$; period 360° or 2π

5. amplitude: 2; period 360° or 2π

6. amplitude: $\frac{2}{3}$; period 360° or 2π

7. amplitude: does not exist; period 180° or π

8. amplitude: does not exist; period 180° or π

9. amplitude: 4; period 180° or π

10. amplitude: 4; period 480° or $\frac{8\pi}{3}$

11. amplitude: does not exist; period 120° or $\frac{2\pi}{3}$

12. amplitude: $\frac{3}{4}$; period 720° or 4π

15. amplitude: 3; period 360° or 2π

16. amplitude: 5; period 360° or 2π
17. amplitude: does not exist; period: $360^\circ$ or $2\pi$

18. amplitude: does not exist; period: $180^\circ$ or $\pi$

19. amplitude: $\frac{1}{5}$; period: $360^\circ$ or $2\pi$

20. amplitude: does not exist; period: $360^\circ$ or $2\pi$

21. amplitude: 1; period $90^\circ$ or $\frac{\pi}{2}$

22. amplitude: 1; period: $180^\circ$ or $\pi$

23. amplitude: does not exist; period: $120^\circ$ or $\frac{2\pi}{3}$

24. amplitude: does not exist; period: $36^\circ$ or $\frac{\pi}{5}$

25. amplitude: does not exist; period: $540^\circ$ or $3\pi$

26. amplitude: does not exist; period: $360^\circ$ or $2\pi$
27. amplitude: 6; period: 540° or $3\pi$

28. amplitude: 3; period: 720° or $4\pi$

29. amplitude: does not exist; period: 720° or $4\pi$

30. amplitude: does not exist; period: 90° or $\frac{\pi}{2}$

31. amplitude: does not exist; period: 180° or π

32. amplitude: $\frac{8}{3}$; period: 600° or $\frac{10\pi}{3}$

33. amplitude: $\frac{3}{2}$; period: 360° or $2\pi$

34. amplitude: $\frac{7}{2}$; period: 180° or $\pi$

38. $f(x) = \cos x$

$f(x) = \cos (-x)$
Page 769, Lesson 14-2
Graphing Calculator Investigation

4. [Graphs showing various functions and transformations.]

[Graphs showing various functions and transformations.]

Pages 774–776, Lesson 14-2

4. [Graph showing a function involving sine.]

5. [Graphs showing functions involving tangent and sine.]

6. [Graph showing a function involving cosine.]

7. [Graph showing a function involving secant.]

8. [Graph showing a function involving cosine.]

9. [Graph showing a function involving secant.]

10. [Graph showing a function involving tangent and sine.]

11. [Graph showing a function involving sine.]

Chapter 14  Additional Answers  811D
27. 
\[ y = \cos \theta - 5 \]

28. 
\[ y = \csc \theta - \frac{1}{2} \]

29. 
\[ y = \frac{1}{3} \sin \theta + \frac{1}{2} \]

30. 
\[ y = 2 \sin \left[ \frac{1}{2} \left( \theta + 45^\circ \right) \right] + 1 \]

33. 1; 2; 120°; 45°

34. -5; 4; 180°; -30°

35. -3.5; does not exist; 720°; -60°

36. 0.75; does not exist; 270°; 90°

37. 1; \( \frac{1}{4} \); 180°; 75°

38. -4; does not exist; 30°; -22.5°

39. 3; 2; π; -\( \frac{\pi}{4} \)
40. 4; does not exist; $6\pi$; $-\frac{2\pi}{3}$

![Graph of $y = 4 + 5 \sec \left(\frac{1}{3}x + \frac{2\pi}{3}\right)$]

The graphs are identical.

41. $y = 3 - \frac{1}{2} \cos \theta$

The graphs are identical.

42. $y = -\sin \left[ \frac{1}{2}(\theta - \frac{\pi}{2}) \right]$

The graphs are identical.

49. Sample answer: You can use changes in amplitude and period along with vertical and horizontal shifts to show an animal population’s starting point and display changes to that population over a period of time. Answers should include the following information.

- The equation shows a rabbit population that begins at 1200, increases to a maximum of 1450 then decreases to a minimum of 950 over a period of 4 years.
- Relative to $y = a \cos bx$, $y = a \cos bx + k$ would have a vertical shift of $k$ units, while $y = a \cos [b(x - h)]$ has a horizontal shift of $h$ units.

Page 781, Lesson 14-3

50. amplitude: does not exist; period: $180^\circ$ or $\pi$

![Graph of $y = \csc 2\theta$]

51. amplitude: 1; period: $120^\circ$ or $\frac{2\pi}{3}$

![Graph of $y = \cos 3\theta$]

52. amplitude: does not exist; period: $36^\circ$ or $\frac{\pi}{5}$

![Graph of $y = \frac{1}{2} \cot 5\theta$]

Page 781, Practice Quiz 1

1. $y = \frac{3}{4} \sin \frac{1}{2} \theta$

![Graph of $y = 2 \cos \left[ \frac{1}{2}(\theta - \frac{\pi}{2}) \right] - 5$]

Pages 784–785, Lesson 14-4

1. $\sin \theta \tan \theta \neq \sec \theta - \cos \theta$

$\sin \theta \tan \theta \neq \frac{1}{\cos \theta} - \cos \theta$

$\sin \theta \tan \theta \neq \frac{1}{\cos \theta} - \frac{\cos^2 \theta}{\cos \theta}$

Multiply by the LCD, $\cos \theta$.

$\sin \theta \tan \theta \neq \frac{1 - \cos^2 \theta}{\cos \theta}$

Subtract.

$\sin \theta \tan \theta \neq \frac{\sin^2 \theta}{\cos \theta}$

Factor.

$\sin \theta \tan \theta = \sin \theta \tan \theta$

$\frac{\sin \theta}{\cos \theta} = \tan \theta$
2. Sample answer: Use various identities, multiply or divide terms to form an equivalent expression, factor, and simplify rational expressions.

3. Sample answer: $\sin^2 \theta = 1 + \cos^2 \theta$; it is not an identity because $\sin^2 \theta = 1 - \cos^2 \theta$.

4. $\tan \theta (\cot \theta + \tan \theta) \neq \sec^2 \theta$

   $1 + \tan^2 \theta \neq \sec^2 \theta$

   $\sec^2 \theta = \sec^2 \theta$

5. $\tan^2 \theta \cos^2 \theta \neq 1 - \cos^2 \theta$

   $\sin^2 \theta \cdot \cos^2 \theta \neq \sin^2 \theta$

   $\sin^2 \theta = \sin^2 \theta$

6. $\frac{\cos^2 \theta}{1 - \sin \theta} \neq 1 + \sin \theta$

   $\frac{1 - \sin^2 \theta}{1 - \sin \theta} \neq 1 + \sin \theta$

   $(1 - \sin \theta)(1 + \sin \theta) \neq 1 + \sin \theta$

   $1 + \sin \theta = 1 + \sin \theta$

7. $\frac{1 + \tan^2 \theta}{\csc^2 \theta} \neq \tan^2 \theta$

   $\frac{\sec^2 \theta}{\csc^2 \theta} \neq \tan^2 \theta$

   $\frac{1}{\cos^2 \theta} \neq \tan^2 \theta$

   $\frac{1}{\sin^2 \theta} \neq \tan^2 \theta$

   $\tan^2 \theta = \tan^2 \theta$

8. $\frac{\sin \theta}{\sec \theta} \neq 1 + \tan \theta + \cot \theta$

   $\frac{\sin \theta}{\sec \theta} \neq 1 + \frac{\sin \theta}{\cos \theta}$

   $\sin \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sin \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sin \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sec \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sec \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sec \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sec \theta \neq \frac{\sin \theta}{\cos \theta}$

   $\sec \theta = \sec \theta$

   $\sec \theta = \sec \theta$

   $\sec \theta = \sec \theta$

   $\sec \theta = \sec \theta$

   $\tan \theta \neq \sec \theta$

9. $\frac{\sec \theta + 1}{\tan \theta} \neq \frac{\sec \theta}{\sec \theta - 1}$

   $\frac{\sec \theta + 1}{\tan \theta} \neq \frac{\sec \theta + 1}{\sec \theta - 1}$

   $\frac{\sec \theta + 1}{\tan \theta} \neq \frac{\sec \theta + 1}{\sec \theta - 1}$

   $\frac{\sec \theta + 1}{\tan \theta} \neq \frac{\sec \theta + 1}{\sec \theta + 1}$

   $\frac{\sec \theta + 1}{\tan \theta} \neq \frac{\sec \theta + 1}{\sec \theta + 1}$

   $\frac{\sec \theta + 1}{\tan \theta} \neq \frac{\sec \theta + 1}{\tan \theta}$

11. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta \neq 1$

   $\cos^2 \theta + \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \cos^2 \theta \neq 1$

   $\cos^2 \theta + \sin^2 \theta \neq 1$

   $1 = 1$

12. $\cot \theta (\cot \theta + \tan \theta) \neq \csc^2 \theta$

   $\cot^2 \theta + \cot \theta \tan \theta \neq \csc^2 \theta$

   $\cot^2 \theta + \frac{\sin \theta}{\cos \theta} \cdot \cot \theta \neq \csc^2 \theta$

   $\cot^2 \theta + 1 \neq \csc^2 \theta$

   $\csc^2 \theta = \csc^2 \theta$

13. $1 + \sec^2 \theta \sin^2 \theta \neq \sec^2 \theta$

   $1 + \frac{1}{\cos^2 \theta} \cdot \sin^2 \theta \neq \sec^2 \theta$

   $1 + \tan^2 \theta \neq \sec^2 \theta$

   $\sec^2 \theta = \sec^2 \theta$

14. $\sin \theta \sec \theta \cot \theta \neq 1$

   $\sin \theta \cdot \frac{1}{\cos \theta} \cdot \frac{\sin \theta}{\sin \theta} \neq 1$

   $1 = 1$

15. $\frac{1 - \cos \theta}{1 + \cos \theta} \neq (\csc \theta - \cot \theta)^2$

   $1 - \cos \theta \neq \csc^2 \theta - 2 \cot \theta \csc \theta + \cot^2 \theta$

   $1 - \cos \theta \neq \frac{1 - \cos \theta}{\sin^2 \theta} - \frac{\cos \theta}{\sin \theta} \cdot \frac{1 + \cos^2 \theta}{\sin^2 \theta}$

   $1 - \cos \theta \neq \frac{1 - \cos \theta}{\sin^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta}$

   $1 - \cos \theta \neq \frac{1 - \cos \theta}{\sin^2 \theta}$

   $1 - \cos \theta \neq \frac{1}{\sin^2 \theta}$

   $1 - \cos \theta = \frac{1}{\sin^2 \theta}$

   $1 - \cos \theta = 1 - \cos \theta$

   $1 - \cos \theta = 1 - \cos \theta$

   $1 - \cos \theta = 1 - \cos \theta$

   $1 - \cos \theta = 1 - \cos \theta$

16. $\frac{1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} \neq \tan \theta - \cot \theta$

   $(1 - \cos^2 \theta) - \cos^2 \theta \neq \frac{\tan \theta - \cot \theta}{\sin \theta \cos \theta}$

   $\frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} \neq \frac{\tan \theta - \cot \theta}{\sin \theta \cos \theta}$

   $\frac{\sin^2 \theta}{\sin \theta \cos \theta} - \frac{\cos^2 \theta}{\sin \theta \cos \theta} \neq \frac{\tan \theta - \cot \theta}{\sin \theta \cos \theta}$

   $\frac{\sin \theta}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \neq \frac{\tan \theta - \cot \theta}{\sin \theta \cos \theta}$

   $\frac{\cos \theta - \sin \theta}{\sin \theta} \neq \frac{\tan \theta - \cot \theta}{\sin \theta \cos \theta}$

   $\tan \theta - \cot \theta = \tan \theta - \cot \theta$

17. $\cot \theta \csc \theta \neq \frac{\cot \theta + \csc \theta}{\sin \theta + \tan \theta}$

   $\cot \theta \csc \theta \neq \frac{1}{\sin \theta + \sin \theta}$

   $\cot \theta \csc \theta \neq \frac{1}{\sin \theta + \sin \theta}$

   $\cot \theta \csc \theta \neq \frac{1}{\sin \theta + \sin \theta}$

   $\cot \theta \csc \theta \neq \frac{\cos \theta + 1}{\sin \theta}$

   $\cot \theta \csc \theta \neq \frac{\cos \theta + 1}{\sin \theta}$

   $\cot \theta \csc \theta \neq \frac{\cos \theta + 1}{\sin \theta}$

   $\cot \theta \csc \theta \neq \frac{\cos \theta + 1}{\sin \theta}$

   $\cot \theta \csc \theta \neq \frac{\cos \theta + 1}{\sin \theta}$

   $\cot \theta \csc \theta = \cot \theta \csc \theta$
18. \[ \sin \theta + \cos \theta = \frac{1 + \tan \theta}{\sec \theta} \]
\[ \sin \theta + \cos \theta = \frac{1 + \sin \theta}{\cos \theta} \]
\[ \sin \theta + \cos \theta = \frac{1}{\cos \theta} \cdot \cos \theta \]
\[ \sin \theta + \cos \theta = \sin \theta + \cos \theta \]

19. \[ \sec \theta - \frac{\sin \theta}{\cos \theta} = \cot \theta \]
\[ \frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta \]
\[ \frac{1}{\sin \theta \cos \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta \]
\[ \frac{1}{\sin \theta \cos \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta \]

20. \[ \sin \theta = \cot \theta \]
\[ \frac{1 - \cos \theta}{\sin \theta} + \frac{1 - \cos \theta}{\sin \theta} = 2 \csc \theta \]
\[ \frac{\sin \theta}{\sin \theta (1 - \cos \theta)} + \frac{\sin \theta}{\sin \theta (1 - \cos \theta)} = 2 \csc \theta \]
\[ \frac{\sin \theta}{\sin \theta (1 - \cos \theta)} = \frac{1}{2} \csc \theta \]
\[ 2 \csc \theta = 2 \csc \theta \]

21. \[ \frac{1 + \sin \theta}{\sin \theta} = \csc^2 \theta \]
\[ \frac{1 + \sin \theta}{\sin \theta} = \csc^2 \theta - 1 \]
\[ \frac{1 + \sin \theta}{\sin \theta} = \csc^2 \theta - 1 \]
\[ \frac{1 + \sin \theta}{\sin \theta} = \frac{1}{\csc \theta} \cdot \csc \theta + 1 \]
\[ \frac{1 + \sin \theta}{\sin \theta} = \frac{1}{\csc \theta} \cdot \csc \theta + 1 \]
\[ \frac{1 + \sin \theta}{\sin \theta} = \frac{1}{\csc \theta} \cdot \csc \theta + 1 \]
\[ \frac{1 + \sin \theta}{\sin \theta} = \frac{1 + \sin \theta}{\sin \theta} \]
\[ \frac{1 + \sin \theta}{\sin \theta} = 1 + \sin \theta \]
\[ \frac{1 + \sin \theta}{\sin \theta} = 1 + \sin \theta \]

22. \[ \frac{1 + \tan \theta}{\sec \theta} = \sin \theta \]
\[ \frac{1 + \sin \theta}{\cos \theta} = \cos \theta \]
\[ \frac{1 + \sin \theta}{\cos \theta} = \cos \theta \]
\[ \frac{\sin \theta + \cos \theta}{\cos \theta} = \sin \theta \]
\[ \frac{\sin \theta + \cos \theta}{\sin \theta} = \cos \theta \]

23. \[ \frac{1}{\sec^2 \theta + \csc^2 \theta} = 1 \]
\[ \cos^2 \theta \sin^2 \theta = 1 \]
\[ 1 = \frac{\tan^2 \theta}{\sec \theta - 1} \]
\[ 1 + \frac{\cos \theta}{\sec \theta - 1} \cdot \sec \theta + 1 \]
\[ 1 + \frac{\cos \theta}{\sec \theta - 1} \cdot \sec \theta + 1 \]
\[ 1 = \frac{\tan^2 \theta}{\sec \theta + 1} \]
\[ 1 + \frac{\cos \theta}{\sec \theta + 1} \cdot \sec \theta + 1 \]
\[ 1 + \frac{\cos \theta}{\sec \theta + 1} \cdot \sec \theta + 1 \]
\[ 1 + \frac{1}{\cos \theta} = 1 + \frac{1}{\cos \theta} \]

25. \[ 1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta \]
\[ (1 - \tan^2 \theta)(1 + \tan^2 \theta) = \sec^2 \theta (2 - \sec^2 \theta) \]
\[ (1 - \sec^2 \theta - 1)(\sec^2 \theta) = (2 - \sec^2 \theta)(\sec^2 \theta) \]
\[ (2 - \sec^2 \theta)(\sec^2 \theta) = (2 - \sec^2 \theta)(\sec^2 \theta) \]

26. \[ \cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta \]
\[ (\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) = \cos^2 \theta - \sin^2 \theta \]
\[ (\cos^2 \theta - \sin^2 \theta) \cdot 1 = \cos^2 \theta - \sin^2 \theta \]
\[ \cos^2 \theta - \sin^2 \theta = \cos^2 \theta - \sin^2 \theta \]

27. \[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]
\[ \frac{1 - \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} \]

28. \[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]

29. \[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
\[ \frac{\cos \theta}{1 + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \]
31. \[ \frac{\tan^2 \theta}{2g \sec^2 \theta} = \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{2g \frac{\cos^2 \theta}{\cos^2 \theta}} = \frac{\sin^2 \theta}{2g} \cdot \frac{\cos^2 \theta}{1} = \frac{\sin^2 \theta}{2g} \]

34. Sample answer: Trigonometric identities are verified in a similar manner to proving theorems in geometry before using them. Answers should include the following.

- The expressions have not yet been shown to be equal, so you could not use the properties of equality on them.
- To show two expressions you must transform one, or both independently.
- Graphing two expressions could result in identical graphs for a set interval, that are different elsewhere.

37. 

38. 

39. 

40. 

41. 

42. 

47. 

48. 

49. 

Pages 788–790, Lesson 14-5

10. \[ \cos (270^\circ - \theta) = \cos 270^\circ \cos \theta + \sin 270^\circ \sin \theta \]

11. \[ \sin \left( \theta + \frac{\pi}{2} \right) = \cos \theta \]

12. \[ \sin (\theta + 30^\circ) + \cos (\theta + 60^\circ) \]

28. \[ \sin (270^\circ - \theta) = \sin 270^\circ \cos \theta - \cos 270^\circ \sin \theta \]

29. \[ \cos (90^\circ + \theta) = \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta \]

30. \[ \cos (90^\circ - \theta) = \cos 90^\circ \cos \theta + \sin 90^\circ \sin \theta \]
31. \[ \sin (90^\circ - \theta) = \cos \theta \]
\[ \sin 90^\circ \cos \theta - \cos 90^\circ \sin \theta = \cos \theta \]
\[ 1 \cdot \cos \theta - 0 \cdot \sin \theta = \cos \theta \]
\[ \cos \theta = \cos \theta \]
32. \[ \sin \left( \theta + \frac{3\pi}{2} \right) = -\cos \theta \]
\[ \sin \theta \cos \frac{3\pi}{2} + \cos \theta \sin \frac{3\pi}{2} = -\cos \theta \]
\[ \sin \theta \cdot 0 + \cos \theta \cdot (-1) = -\cos \theta \]
\[ 0 + (-\cos \theta) = -\cos \theta \]
\[ -\cos \theta = -\cos \theta \]
33. \[ \cos (\pi - \theta) = -\cos \theta \]
\[ \cos \pi \cos \theta + \sin \pi \sin \theta = -\cos \theta \]
\[ -1 \cdot \cos \theta = 0 \cdot \sin \theta = -\cos \theta \]
\[ -\cos \theta = -\cos \theta \]
34. \[ \cos (2\pi + \theta) = \cos \theta \]
\[ \cos 2\pi \cos \theta - [\sin 2\pi \sin \theta] = \cos \theta \]
\[ 1 \cdot \cos \theta - 0 \cdot \sin \theta = \cos \theta \]
\[ 1 \cdot \cos \theta = 0 \cdot \cos \theta \]
\[ \cos \theta = \cos \theta \]
35. \[ \sin (\pi - \theta) = \sin \theta \]
\[ \sin \pi \cos \theta - [\cos \pi \sin \theta] = \sin \theta \]
\[ 0 \cdot \cos \theta - [-1 \cdot \sin \theta] = \sin \theta \]
\[ 0 - [-\sin \theta] = \sin \theta \]
\[ \sin \theta = \sin \theta \]
36. \[ \sin (60^\circ + \theta) + \sin (60^\circ - \theta) = \sin \left( \frac{\sqrt{3}}{2} \right) \]
\[ \cos \theta + \frac{1}{2} \sin \theta + \sqrt{3} \cos \theta - \frac{1}{2} \sin \theta \]
\[ = \sqrt{3} \cos \theta \]
37. \[ \sin \left( \theta + \frac{\pi}{3} \right) - \cos \left( \theta + \frac{\pi}{6} \right) = \sin \theta \]
\[ \cos \frac{\pi}{3} \cos \theta + \sin \frac{\pi}{3} \sin \theta - \cos \theta \cos \frac{\pi}{6} + \sin \theta \sin \frac{\pi}{6} \]
\[ = \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta - \sqrt{3} \cos \theta + \frac{1}{2} \sin \theta \]
\[ = \frac{1}{2} \sin \theta + \frac{1}{2} \sin \theta \]
\[ = \sin \theta \]
38. \[ \sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta \]
\[ \sin (\alpha \cos \beta + \cos \alpha \sin \beta) \sin (\alpha \cos \beta - \cos \alpha \sin \beta) \]
\[ \sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta \]
\[ \sin^2 \alpha (1 - \sin^2 \beta) - (1 - \sin^2 \alpha) \sin^2 \beta \]
\[ \sin^2 \alpha - \sin^2 \alpha \sin^2 \beta - \sin^2 \beta + \sin^2 \alpha \sin^2 \beta \]
\[ = \cos^2 \alpha - \sin^2 \beta \]
39. \[ \cos (\alpha + \beta) \]
\[ \cos (\alpha + \beta) = \frac{1 - \tan \theta \tan \beta}{\sec \alpha \sec \beta} \]
\[ \cos (\alpha + \beta) \]
\[ \cos (\alpha + \beta) = \frac{1 - \sin \alpha \sin \beta}{\cos \alpha \cdot \cos \beta} \]
32. \[2 \cos^2 \frac{x}{2} \pm \frac{1}{2} = 1 + \cos x\]
\[2 \left( \pm \sqrt{\frac{1 + \cos x}{2}} \right)^2 \pm \frac{1}{2} = 1 + \cos x\]
\[2 \left( \frac{1 + \cos x}{2} \right)^2 \pm \frac{1}{2} = 1 + \cos x\]
\[1 + \cos x = 1 + \cos x\]

33. \[\sin^4 x - \cos^4 x \pm 2 \sin^2 x - 1\]
\[(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) \pm 2 \sin^2 x - 1\]
\[(\sin^2 x - \cos^2 x) \cdot 1 \pm 2 \sin^2 x - 1\]
\[(\sin^2 x - (1 - \sin^2 x)) \cdot 1 \pm 2 \sin^2 x - 1\]
\[\sin^2 x - 1 + \sin^2 x \pm 2 \sin^2 x - 1\]
\[2 \sin^2 x - 1 = 2 \sin^2 x - 1\]

34. \[\sin^2 x \pm \frac{1}{2} (1 - \cos 2x)\]
\[\sin^2 x \pm \frac{1}{2} [1 - (1 - 2 \sin^2 x)]\]
\[\sin^2 x \pm \frac{1}{2} (2 \sin^2 x)\]
\[\sin^2 x = \sin^2 x\]

35. \[\tan^2 \frac{x}{2} \pm 1 - \cos x\]
\[\sin^2 \frac{x}{2} \pm 1 + \cos x\]
\[\frac{\pm \sqrt{1 - \cos x}}{2} \pm \frac{1}{2} = 1 - \cos x\]
\[\frac{\pm \sqrt{1 + \cos x}}{2} \pm \frac{1}{2} = 1 + \cos x\]
\[1 - \cos x = 1 - \cos x\]

36. \[-\frac{1}{\sin x \cos x} \pm \frac{\cos x}{\sin x} \pm \tan x\]
\[\frac{1 - \cos^2 x}{\sin x \cos x} \pm \frac{\tan x}{\sin x \cos x} \pm \tan x\]
\[\frac{\sin x}{\cos x} \pm \frac{\tan x}{\cos x} \pm \tan x\]
\[\tan x = \tan x\]

40. \[\frac{2}{g} v^2 (\tan \theta - \tan \theta \sin^2 \theta) \pm \frac{2}{g} v^2 \tan \theta (1 - \sin^2 \theta)\]
\[\frac{2}{g} v^2 \tan \theta \cos^2 \theta\]
\[\frac{2}{g} v^2 \sin \theta \cos \theta\]
\[\frac{v^2}{g} \sin 2\theta\]

42. Sample answer:
They all have the same shape and are vertical translations of each other.

43. The maxima occur at \[\pm \frac{\pi}{2}\] and \[\pm \frac{3\pi}{2}\]. The minima occur at \[x = 0, \pm \pi, \text{and } \pm 2\pi\].
5. \( \cos \left( \frac{3\pi}{2} - \theta \right) = -\sin \theta \)
\[
\cos \frac{3\pi}{2} \cos \theta + \sin \frac{3\pi}{2} \sin \theta = -\sin \theta
\]
\[
0 + (-1 \cdot \sin \theta) = -\sin \theta
\]
\[
-\sin \theta = -\sin \theta
\]
6. \( \sin (\theta + 30^\circ) + \cos (\theta + 60^\circ) \)
\[
= \left( \sin \theta \cos 30^\circ + \cos \theta \sin 30^\circ \right) +
\left( \cos \theta \cos 60^\circ - \sin \theta \sin 60^\circ \right)
\]
\[
= \left( \frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta \right) + \left( \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta \right)
\]
\[
= \frac{1}{2} \cos \theta + \frac{1}{2} \cos \theta
\]
\[
= \cos \theta
\]

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Graphing Calculator Investigation

1. [Graph of a function from 0 to 360 degrees with a scale of 1 by 5 and 45 units by 2 units]

2. [Graph of a function from 0 to 360 degrees with a scale of 1 by 5 and 45 units by 2 units]

3. [Graph of a function from 0 to 360 degrees with a scale of 1 by 5 and 45 units by 2 units]

4. [Graph of a function from -720 to 720 degrees with a scale of 1 by 5 and 45 units by 2 units]

5. [Graph of a function from 0 to 360 degrees with a scale of 1 by 5 and 45 units by 2 units]

6. [Graph of a function from -360 to 360 degrees with a scale of 1 by 5 and 45 units by 2 units]

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35. \( 0 + 2k\pi, \frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi \text{ or } 0^\circ + k \cdot 360^\circ, \)
\( 90^\circ + k \cdot 360^\circ, 270^\circ + k \cdot 360^\circ \)

36. \( \frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi \text{ or } 210^\circ + k \cdot 360^\circ, \)
\( 330^\circ + k \cdot 360^\circ \)

37. \( 0 + k\pi \text{ or } 0^\circ + k \cdot 180^\circ \)

38. \( \frac{\pi}{2} + k\pi, \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi \text{ or } 90^\circ + k \cdot 180^\circ, \)
\( 120^\circ + k \cdot 360^\circ, 240^\circ + k \cdot 360^\circ \)

39. \( 0 + 2k\pi, \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi \text{ or } 0^\circ + k \cdot 360^\circ, \)
\( 60^\circ + k \cdot 360^\circ, 300^\circ + k \cdot 360^\circ \)

40. \( \frac{\pi}{2} + 4k\pi \text{ or } 90^\circ + k \cdot 720^\circ \)

41. \( y = \frac{3}{2} + \frac{3}{2} \sin (\pi t) \)
13. amplitude: does not exist; period: $540^\circ$ or $3\pi$

14. amplitude: does not exist; period: $45^\circ$ or $\frac{\pi}{4}$

34. \[ \cos (90^\circ + \theta) \overset{?}{=} -\sin \theta \]
\[ \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta \overset{?}{=} -\sin \theta \]
\[ 0 \cdot \cos \theta - 1 \cdot \sin \theta \overset{?}{=} -\sin \theta \]
\[ -\sin \theta = -\sin \theta \]

35. \[ \sin (30^\circ - \theta) \overset{?}{=} \cos (60^\circ + \theta) \]
\[ \sin 30^\circ \cos \theta - \cos 30^\circ \sin \theta \overset{?}{=} \cos 60^\circ \cos \theta - \sin 60^\circ \sin \theta \]
\[ \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta \]

36. \[ \sin (\theta + \pi) \overset{?}{=} -\sin \theta \]
\[ \sin \theta \cos \pi + \cos \theta \sin \pi \overset{?}{=} -\sin \theta \]
\[ (\sin \theta)(-1) + (\cos \theta)(0) \overset{?}{=} -\sin \theta \]
\[ -\sin \theta = -\sin \theta \]

37. \[ -\cos \theta \overset{?}{=} \cos (\pi + \theta) \]
\[ -\cos \theta \overset{?}{=} \cos \pi \cos \theta - \sin \pi \sin \theta \]
\[ -\cos \theta \overset{?}{=} -1 \cdot \cos \theta - 0 \cdot \sin \theta \]
\[ -\cos \theta = -\cos \theta \]

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9. \[ (\sin \theta - \cos \theta)^2 \overset{?}{=} 1 - 2\sin \theta \cos \theta \]
\[ \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta \overset{?}{=} 1 - 2\sin \theta \]
\[ (\sin^2 \theta + \cos^2 \theta) - 2 \sin \theta \cos \theta \overset{?}{=} 1 - 2\sin \theta \]
\[ 1 - \sin 2\theta = 1 - 2\sin \theta \]

10. \[ \frac{\cos \theta}{1 - \sin^2 \theta} \overset{?}{=} \sec \theta \]
\[ \frac{\cos \theta}{\cos^2 \theta} \overset{?}{=} \sec \theta \]
\[ \frac{1}{\cos \theta} \overset{?}{=} \sec \theta \]
\[ \sec \theta = \sec \theta \]

11. \[ \frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \overset{?}{=} \cot \theta \]
\[ \frac{1}{\sin \theta \cos \theta} - \frac{\sin \theta}{\cos \theta} \overset{?}{=} \cot \theta \]
\[ \frac{1}{\sin \theta \cos \theta} - \frac{\sin^2 \theta}{\sin \theta \cos \theta} \overset{?}{=} \cot \theta \]
\[ \frac{\cos^2 \theta}{\sin \theta \cos \theta} \overset{?}{=} \cot \theta \]
\[ \cot \theta = \cot \theta \]

12. \[ \frac{1 + \tan^2 \theta}{\cos^2 \theta} \overset{?}{=} \sec^4 \theta \]
\[ \frac{\sec^2 \theta}{\cos^2 \theta} \overset{?}{=} \sec^4 \theta \]
\[ \sec^2 \theta \sec^2 \theta \overset{?}{=} \sec^4 \theta \]
\[ \sec^4 \theta = \sec^4 \theta \]